The Emergence and Future of Central Counterparties*

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Abstract

Forward, derivative and swap markets have taken on a large and ever increasing role in allocating aggregate risk intertemporally, at the cost of introducing counterparty risk. Over time, a mechanism known as a Central Counterparty (CCP) has been introduced to provide insurance against the default risk prevalent in, e.g. futures and derivatives markets. We propose a model that can explain why CCPs emerged on centralized exchange and explain how to design their policies. We then use this model to analyze the effects of introducing mandatory CCP clearing on trades conducted over-the-counter. While the nature of OTC trades limits the scope of CCP clearing, we still find that it has a role which goes beyond the simple netting of positions.

Very Preliminary and Incomplete

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1 Introduction

Forward, derivative and swap markets have taken on a large and ever increasing role in allocating aggregate risk intertemporally. Over centuries, forward markets have dealt with default risk by first relying on repeated transactions and reputation, and then on collateral and more modern techniques offered by clearinghouses, known as novation and mutualization. But in the wake of the last financial crisis, the current tools have often been deemed inappropriate, especially for trades that occurred over-the-counter (OTC).

In particular, policy makers and market participants have recently advocated for the clearing of trades by a central counterparty (CCP), arguing that this is a necessary element of an efficient market structure. What are CCPs and what do they do? What is the role of collateral, novation and mutualization in the management of default risk? Is CCP clearing appropriate for any market structure, centralized exchange or OTC trades? How should CCP-clearing be designed on an OTC market?

In this paper we present a framework to propose an answer to these and other questions related to CCP.\(^1\) Our model is inspired by the history of the Chicago Mercantile Exchange as related in Kroszner (1999). Risk averse farmers have to decide on how much wheat to grow, before they know the demand for wheat. The demand for wheat (by bakers) is uncertain, however, due to an aggregate demand shock. Therefore farmers revenue, and so consumption, is uncertain. To insure against this income risk, farmers can trade forward contracts. This is the promise to deliver some wheat at a given price. When there is no risk of default, a forward market insures farmers.

However, forward contracts do not work very well when there is a risk of default. In our model, farmers each have to deal with a single baker that can go bust. If a farmer contracted with a now bankrupt baker, he can sell his wheat on the spot market, but at the spot price. So, with no other mechanism in place, a farmer trading forward will still face two types of risk: first the risk that his counterparty goes bust, and second, the resulting price risk of selling wheat in the spot market.

We then study three risk management tools adding them in succession: collateral, novation and mutualization of losses. We model collateral as an asset that bakers can produce on demand. Collateral is costly, but can be seized if a baker goes bust. As it is costly to produce, however, requiring collateral reduces the attractiveness of a forward contract for

\(^1\) We do not look at the services of netting and information provision. While important, such services do not have to be carried out within a central counterparty structure. In particular, see Koepppl, Monnet and Temzelides (2008) for analyzing information issues in clearing OTC.
bakers and therefore the forward contract price. In turn, although farmers are somewhat insured against default, they consume less when there is no default. This effect implies that full collateralization is not optimal, so that farmers still face some uncertainty.

We then introduce novation. This is the process where a third party becomes the buyer to every seller and the seller to every buyer of a forward contract, while guaranteeing the terms of the agreed trade. As this party is the counterparty that is common (or central) to all traders, it is referred to as the central counterparty, or CCP. Through novation, a CCP becomes the seller of wheat to all bakers. Hence, while some bakers still go bust, the CCP does not face a default risk as its portfolio is completely diversified. Therefore, novation eliminates counterparty risk for farmers. Novation is a costless insurance mechanism against counterparty risk, while collateral is costly. So, with novation, the optimal collateral policy is to require no collateral. While this seems extreme, this is due to the exogeneity of default. Were defaults endogenous (i.e. strategic), we show it would be optimal to require some collateral.

While novation eliminates counterparty risk, novation does not eliminate the price risk (or the replacement cost risk): when bakers go bust, the CCP still has to sell the wheat on the spot market at the given price. Therefore its revenue is depending on the spot price. As the promised payment to farmers depend on the CCP revenue, farmers still face the price risk to some extent.

The mutualization of losses (transfers) takes care of the remaining price risk. Assuming a “survivor-pays-rule”, a CCP can impose an additional payment from bakers who did not go bust when wheat is cheap. To make it worth for bakers, the CCP makes them a transfer when wheat is expensive. In this way, the CCP can smooth its revenue which then does not depend on wheat price. In the end, this acts as an insurance scheme for farmers. It results that with novation and mutualization, the efficient allocation where farmers are fully insured is achieved.

We can draw several implications for the historic emergence of forward markets and risk management tools. When collateral is very expensive, it is still optimal to trade forward even without collateral, provided that the default risk is not too high. Hence, historically local forward markets could develop whenever there was good information about the quality of counterparties, even though collateral (and, hence, enforcement) was scarce. As costs of collateral decreased, forward markets could expand geographically, even though that meant a dilution in the quality of counterparties. Finally, as new technology arrived that made more sophisticated services such as a CCP available, markets could even tolerate a higher
level of aggregate default risk, while reducing the cost of pledging collateral.\footnote{Monnet (2009) describes the historic evolution of CCPs and their services.}

In the second part of the paper, we study the future of CCP services, if some of the current proposals to make such services mandatory take effect. We introduce the possibility for farmers and bakers to trade directly and bilaterally over the counter. Trading is bilateral because bakers now have preferences over very special types of wheat as well as standard forms of wheat. Bakers can have different values for their special wheat too. Special wheat can only be traded over the counter as it has to be taylored to the specific needs of the baker.

Then, farmers have the choice to produce either standard wheat with a forward contract or special wheat over the counter. If they trade OTC, they are matched with a baker. The cost for farmers of trading OTC is that special wheat has no value on the spot market in case bakers default. The benefit of trading OTC is that the farmer makes a take-it or leave-it offer to the baker and therefore can extract all the baker’s surplus. This means that farmers will only trade OTC with bakers if the OTC contract generates enough surplus, i.e. if baker’s valuation of their special wheat is high enough. Therefore, this generates a threshold valuation below which farmers prefer to trade on the forward market.

There are at least three sources of inefficiencies on the OTC market. First, each farmer is matched with a single baker. Therefore, each farmer’s consumption is constrained by what this baker is willing to produce. Second, farmers face the risk that the baker defaults. Third, the terms of trade are set by farmers as they make a take-it or leave-it offer. We show that a CCP can alleviate the first two inefficiencies. We find that novation alleviates the counterparty risk, as it would on the forward market. The difference is that, as OTC contracts are not fungible, the CCP is subject to a relatively large replacement risk. By affecting the attractiveness of some contracts, we find that well designed mutualization can reduce the replacement cost risk and get closer to the first best level of consumption for farmers. When we solve for the best mutualization scheme, we show that the CCP’s optimal default exposure should be skewed toward socially valuable contracts, by having more contracts with high valuation. Mutualization can achieve this by taxing the surplus from low valuation matches and subsidizing the one for high valuation matches. Hence, the optimal mutualization scheme achieves an efficient redistribution of default exposure given the terms of trades. However, those are influenced by the bargaining protocol used in OTC trades. Since the CCP takes the terms of the contract as given, mutualization comes short of achieving the efficient distribution of default exposures.

The bottom line is that CCP can go a long way to insure against the risk contained in OTC
trades, in spite of the fact that OTC trades are non-fungible. The bargaining protocol is the source of the remaining inefficiency and there is nothing a CCP can do about it. In a last section, we offer some thoughts on the role of dealers in OTC markets.

2 The Environment

The economy has two periods. There are two goods, a general asset that is storable and wheat. There are two types of agents, a measure 1 of farmers and a measure \(1/(1 - \delta)\) of bakers. Agents do not discount the future.

Farmers consume the asset and can produce wheat. However, it takes time to produce it, as each hour of work at \(t = 1\) returns 1 unit of wheat at \(t = 2\). Bakers die with probability \(\delta\) at the end of period 1, so that there is a measure 1 of bakers alive in period 2. Bakers consume both wheat and the asset, but can only produce the asset. Producing the asset in period 1 is more costly, as it takes \(\mu z\) hours of work in period 1 to produce \(z\) units with \(\mu > 1\) but only \(z\) hours of work in period 2.

Preferences are given by a quasi-linear utility function. Farmers value the consumption of \(x\) units of the asset when they produced \(q\) units of wheat according to the utility function \(u(x, q) = \log(x) - q\). Bakers value the production of \(z_1\) units of the asset in period 1, \(z_2\) units in period 2, while consuming \(y\) units of wheat according to the expected utility function \(v(y, z) = (1 - \delta) [\theta_i \log(y) - z_2] - \mu z_1\). The preference shock \(\theta_i\) is given by \(\theta_i = \theta + \varepsilon_i\), where \(\varepsilon_i\) is iid according to a distribution \(G\) and has mean 0, while \(\theta\) is an aggregate shock drawn from a distribution \(F\) and has mean 1.

This environment captures the fundamental frictions that will allow us to endogenize the need for forward contracts. A risk averse farmer would rather be paid early for his work, as the shock \(\theta\) affects the value of his production. However, bakers would prefer to postpone their payments. Ignoring any default risk, a futures contract can guarantee a certain payment to farmers, while allowing bakers to pay late. Before analyzing the contractual environment, we solve for the efficient allocation.

\(^3\)We adopt the convention that \(z < 0\) means the asset is consumed by the baker.
3 The First Best Allocation

We consider the problem of a planner who maximizes a social welfare function that gives equal weight to both types of agents. Then the first best symmetric allocation \((x^* (\theta), y^* (\theta))\) solves,

\[
\max_{x,y,q} \int \log (x_1 + x_2 (\theta)) - q + \theta_i \log (y (\theta_i; \theta)) - x_2 (\theta_i; \theta) - \mu (x_1 / 1 - \delta) dG (\varepsilon_i) dF (\theta)
\]

\[s.t. \int y (\theta_i) dG (\theta_i) \leq q, \quad \text{and} \quad x_2 (\theta) \leq \int x_2 (\theta_i; \theta) dG (\theta_i) \quad \text{for all} \ \theta\]

Here \(x_t (\theta)\) is the farmer’s consumption in period \(t\), while \(y (\theta_i; \theta)\) is the baker’s consumption of wheat in period 2 when the aggregate preference shock is \(\theta\) and the idiosyncratic shock is \(\theta_i\). Naturally, all allocations in period 1 are independent of the realization of \(\theta\). Note that it is cheaper to produce the farmers’ consumption good in period 2, and that farmers are indifferent between consuming in period 1 or 2. Therefore, \(x_1^* = 0\). The first order conditions with respect to \(x_2 (\theta), q\) and \(y (\theta_i; \theta)\) and the complementary slackness condition give us the efficient solution \(q^* = 1\) and \(y^* (\theta_i; \theta) = \theta_i / \theta\), while \(x_2 (\theta) = 1\) and \(x_2^* (\theta_i; \theta)\) is indeterminate. Concavity implies this solution is unique (up to the distribution of the production of the asset among bakers).

To sum up, the first best solution completely insures farmers against the aggregate shock. Also it equates marginal utility across all agents of the same type. The investment in wheat in period 1 is just sufficient to give a consumption of \(\theta_i / \theta\) unit to a baker with shock \(\theta_i\). Each baker still alive produces 1 unit of assets to ensure each farmer can consume one unit.

4 Implementation

In this section we look at different market mechanisms and how they fare relative to the first best allocation. We find it useful to move from the coarsest mechanisms (spot trading) and then refine it along the way. This allows us to clearly illustrate which aspects of the trading environment gets us closer to the first best. We start from a simple spot market. Then we allow trading in a forward market and later we introduce a CCP.
4.1 Spot market

First we look at the agents’ decision when they enter the market for wheat in period 2. We call this market the “spot market” since in this market, assets are traded with immediate settlement.

We first consider the value \( U(R) \) for a farmer of entering the spot market with a portfolio worth \( R \) assets:

\[
U(R) = \max_x x \log(x) \quad \text{s.t. } x \leq R.
\]

Let us now consider the value \( V(R) \) for a baker of entering the spot market with a portfolio worth \( R \) assets. Then

\[
V(R) = \max_{y,z} \theta_i \log(y) - z \quad \text{s.t. } py \leq z + R.
\]

As the budget constraint will always bind, we have

\[
V(R) = R + \max_y \left[ \theta_i \log(y) - py \right].
\] (1)

The first order condition gives

\[
y(\theta_i) = \theta_i/p.
\] (2)

As expected, bakers’ decision on the spot market is independent of their revenue (a feature of linearity). Hence, in period 1 bakers will just maximize \( R \).

Suppose now there is no market in period 1. Then the revenue of bakers is \( R = 0 \). Farmers choose their initial investment in wheat to maximize their expected payoff, given that \( p(\theta) \) is the spot market price of wheat in period 2 in state \( \theta \). If a farmer invests \( q \) units of wheat in period 1, he consumes all of his revenue in period 2, which is \( p(\theta)q \) in state \( \theta \). Therefore, a farmer’s problem in period 1 is

\[
\max_q -q + \int \log(p(\theta)q) \, dF(\theta).
\]

The first order condition gives us

\[
q = 1.
\] (3)

Finally, the market clearing condition requires \( \int y(\theta_i) \, dG(\epsilon_i) = q \). Using (2) and (3), the equilibrium spot price is state dependent and equals

\[
p = \theta.
\] (4)
When the market structure consists of only a spot market, the allocation differs from the first best in two aspects. First, the revenue of farmers varies with the aggregate demand shock for wheat. When the price of wheat is low, farmers do not get much revenue from their production and as a result, their consumption level is low (and inversely). Second, the bakers’ allocation is also affected by the price in period 2, as their production of the asset will vary with their preference shock. When the demand for wheat increases, the price is higher and bakers have to work more to consume the same amount. Finally, bakers’ consumption and expected production achieve their first best level, as \( y(\theta_i) = \theta_i / \theta \), while \( \int p y(\theta_i) \, dG(\varepsilon_i) = 1 \). Replacing the equilibrium allocation, welfare ex-ante is

\[
W = \int \log(\theta) - 1 + \left[ \theta_i \log\left(\frac{\theta_i}{\theta}\right) - \theta_i \right] \, dG(\varepsilon_i) \, dF(\theta)
\]

(5)

### 4.2 Forward Contracts

We now assume that on top of the spot market in period 2, there is also a forward market open in period 1, where agents trade forward contracts. A forward contract is a promise to deliver a unit of wheat at date 2 at a price \( p_f \).\(^4\) Without loss of generality, we assume \( p_f \) is state independent so as to insure farmers against the aggregate shock.\(^5\) We let \( q_1 \) denote the number of contracts sold by farmers and \( q_2 \) the number of contracts bought by bakers.

We make an additional assumption: a farmer cannot diversify the risk that a baker dies so that if a farmer trades on the forward market, then he can only trade with a single baker. To make things more palatable, we assume that only a random measure 1 of bakers are allowed to participate in the forward market. Then, as bakers die with probability \( \delta \geq 0 \), it is well possible that a farmer is unable to get the promised payment. In this case, we say that this farmer faces default. Default is a possibly bad outcome, but farmers who faced default can always sell their wheat on the spot market.\(^6\) So, if the price of wheat is high, farmers will be better off if they face default. Bakers who are absent from the forward market and still alive in period 2 can still acquire wheat on the spot market.

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\(^4\)Alternatively, \( p_f \) is the price of a forward contract that promise to deliver 1 unit of wheat at date 2.

\(^5\)Since farmers are risk averse and bakers are risk neutral, the certainty equivalent price is always preferred to any state dependent price schedule.

\(^6\)We chose to have an exogenous default rate because endogenous default introduces intricacies that would blur the main message of this paper. The case with endogenous default is available from the authors upon request.
4.2.1 Equilibrium With A Forward Market

In period 2, a farmer who does not face default has a revenue \( p_f q_1 \). To the contrary, a farmer who faces default has to sell his wheat on the spot market and has revenue \( p(\theta) q_1 \). In period 1, a farmer then solves the following problem

\[
\max_{q_1} -q_1 + (1 - \delta) \log(p_f q_1) + \delta \int \log(p(\theta) q_1) dF(\theta)
\]

(6)

with first order condition \( q_1 = 1 \). Conveniently, our logarithmic utility assumption implies that the production of wheat is not influenced by the default risk.\(^7\) Notwithstanding, farmers do not share the default burden equally. Farmers facing default assume the entire default loss. Their consumption will depend not only on whether they faced default, but also on the aggregate shock \( \theta \). Hence, there are two potential benefits for them to manage the default risk.

The baker who is still alive in period 2 has a claim to \( q_2 \) units of wheat subject to paying the forward price \( p_f \). However, since he can also sell these units on the spot market, his revenue is \( (p(\theta) - p_f) q_2 \). Therefore, from (1) in period 1 bakers choose \( q_2 \) to solve,

\[
\max_{q_2} \int (p(\theta) - p_f) q_2 dF(\theta)
\]

and in period 2 they choose \( y_i \) to solve (2). Using the market clearing condition in the spot market, the equilibrium forward price has to satisfy

\[
p_f = \int p(\theta) dF(\theta) = \int \frac{\theta}{q_1} dF(\theta) = 1.
\]

(7)

Hence, no-arbitrage implies that the forward price equals the expected spot price. Therefore, bakers are indifferent (ex-ante) between trading spot and trading forward, as they do not expect to make a profit from trading forward. When default is exogenous, the forward price is independent of the probability of default because bakers only pay if they are alive, whether they trade spot or forward. Risk averse farmers will prefer to sell their entire production of wheat forward. Since only a measure 1 of bakers can participate in the forward market, market clearing implies

\[
q_2 = q_1.
\]

(8)

Notice that even if farmers sell all their wheat forward, this does not mean that there is no supply of wheat on the spot market in period 2. First, if there is default, some farmers will

\(^7\) The income and substitution effects cancel each other.
get stuck with their wheat and will want to sell it on the forward market. Second, even if there is no default, so that farmers hand all their production to bakers in period 2, bakers still face an idiosyncratic risk that they trade off in the spot market.

Using the expression for the prices and quantities, we obtain that welfare is

\[ W_f = \delta \int \log(\theta) dF(\theta) - 1 + \int \theta_i \log \left( \frac{\theta_i}{\theta} \right) - \theta_i dG(\varepsilon_i) dF(\theta) \]  

(9)

When there is no default \( \delta = 0 \), forward contracts achieve the efficient allocation, as farmers are able to insure completely against the aggregate price shock. As a result \( W_f > W \) from (5) and (9). When \( \delta > 0 \), while farmers’ initial investment remains constant, default introduces consumption risk in the second period. Still, comparing (5) and (9), a forward market followed by a spot market does better than a spot market alone when there is default.

**Proposition 1.** A spot and forward markets (Pareto) dominate a spot market alone.

While shallow, this result is important as it tells us that a forward contract is needed in the current environment, in the sense that it (weakly) increases welfare for all agents. Next, we look at the risk management tools that can improve the performance of the forward market when there is default. We in turn discuss the difference between collateral, novation through a central counterparty and mutualization, and then compare these policies to trading spot.

### 4.3 Forward Contracts And Collateral

Default is an issue as it unambiguously lowers ex-ante welfare. In this section, we analyze whether the use of collateral can reduce the severity of the default problem. In our environment, there is no difference between pledging collateral and pre-paying, hence we will use both terms interchangeably.\(^8\) Farmers can require some of the baker’s asset in period 1 as collateral. While it insures some consumption in case the baker defaults, collateral is inefficient, as the asset is more costly to produce in period 1. We let \( k \) denote the required prepayment for each unit bought. There is no exposure to default if bakers pay the full price ex-ante, \( k = p_f^k \); while there is some exposure otherwise.

We solve for the optimal collateral policy by first considering how it impacts the terms of trade. Whenever \( k \leq p_f^k \), the price of the forward contract is still set by bakers’ optimization

\(^8\) As far as there is partial prepayment, one can think about settling an obligation through netting collateral with the final payment.
problem,
\[
\max_{q_2} -\mu k q_2 + (1 - \delta) \int V \left( (p(\theta) + k - p^*_f q_2) q_2 \right) dF(\theta)
\]
and, by no arbitrage, we obtain the following pricing relationship
\[
p^*_f + \left( \frac{\mu}{1 - \delta} - 1 \right) k = \int p(\theta) dF(\theta). \tag{10}
\]
Once again, note that conveniently \( q_1 = 1 \). Replacing \( p(\theta) \) from (4) in (10) while defining \( \mu(\delta) = \mu / (1 - \delta) > 1 \), we obtain that
\[
p^*_f = 1 + (1 - \mu(\delta)) k. \tag{11}
\]
Note in particular that \( p^*_f < p_f \) for any \( k \). The collateralized forward contract price drops because bakers have to be compensated for the cost of pledging collateral ex-ante, as they could always buy goods in the spot market. There are two reasons why collateral is costly here: first, the direct cost \( \mu > 1 \) of producing the asset early, and second, the indirect cost that collateral is lost if the baker dies. Since they have to be indifferent between buying forward or spot, the forward price needs to fall. Farmers are hit with a lower forward contract price, and so a lower revenue. Still, they benefit from requiring collateral as this insures a portion of their consumption. As the collateral requirement increases, the forward contract price falls and so does farmers’ consumption.

Farmers will choose the level of collateral that maximizes their expected utility. We obtain the following result. The proof is in the Appendix.

**Proposition 2.** The optimal collateral policy is \( k^* = 0 \) if and only if \( \mu \geq \mu(\delta) > 1 \), where \( \partial \mu / \partial \delta > 0 \) for all \( \delta > 0 \). In addition \( k^* < p^*_f \) so that some default loss is always optimal.

Intuitively, it is optimal to use collateral whenever the cost of collateral is low or, given a cost of collateral, whenever the default problem is sufficiently acute, i.e. \( \delta \) is sufficiently high. Maybe surprisingly, farmers never fully collateralize their trade. The intuition is simple: In case of default, farmers can still sell their production spot in period 2. If farmers were to fully collateralize and require \( k = p_f \) they would enjoy too much consumption in default states, at the expense of lower consumption in non default states. Therefore, they prefer to undercollateralize their exposures.

Finally, it is clear that farmers will prefer trading on the forward market, requiring collateral than trading spot. The reason is that no collateral requirement is always an option, but we know that this gives a higher payoff to farmers than trading spot.
To sum up, a forward market with no collateral is best when the cost $\mu$ of acquiring assets is high. As the collateral costs drop, it is optimal to ask for collateral. This is interesting in a historical perspective. Although collateral was expensive, low counterparty risk – in our case a low $\delta$ – was enough to see the emergence of forward markets. Over time, collateral costs have decreased considerably. Additional collateral requirements were also put in place. Our theory says this is optimal. Also, as markets expanded geographically, it became harder to control the quality of counterparties, thus imposing additional collateral requirements. Still, we observe that contracts are undercollateralized, in the sense that some exposure remains. But we show here that this is optimal. Needless to say, this is important, as it means that even when we allow collateral, a forward contract will not achieve the first best allocation. We introduce next a central counterparty.

4.4 Implementation With A Central Counterparty

We introduce now the notions of a CCP. This is a third party such as a clearing agent or a clearinghouse.\textsuperscript{9} All trades are cleared through the CCP, and while the CCP takes the terms of trades as given, the CCP can affect them by modifying the trading environment, such as the collateral requirements.

4.4.1 Novation

Novation is a mechanism whereby the CCP becomes the buyer to every seller and the seller to every buyer. In our environment, this means that farmers are now facing only the clearing agent in period 2 when settling forward contracts. Also, the clearing agent can require additional collateral before clearing a contract. We denote it by $k$. In period 2, the clearing agent pays farmers and bakers directly. Therefore, the clearing agent can pay all farmers the same amount \textit{independently} of whether their original counterparty defaulted or not. As we will show, novation is one mechanism through which the clearing agent offers risk sharing to farmers.\textsuperscript{10}

\textsuperscript{9}In its most primitive function, it could simply be a collateral storage facility. A collateral storage facility might be necessary if farmers cannot commit to make the necessary investment in wheat when they sell the forward contracts and receive collateral. The third party then holds the collateral in escrow until the quantity of wheat promised forward is released to the baker. Hence, with a two-sided default problem, a neutral third party storing collateral is essential for managing default risk, as has been pointed out in a companion paper (see Koepl and Monnet, 2008). Rather than making this notion precise in this framework, we abstract from this issue and assume that farmers can perfectly commit.

\textsuperscript{10}Novation is \textit{not a guarantee}. In order for it to be a guarantee, we would have to require that the CCP satisfies a solvency constraint. In other words, the CCP would guarantee to settle all trades at the price $p_f^\ast$.
Given a collateral policy \( k \), we denote the price of a forward contract with novation by \( p_{nf}^n(k) \). Once contracts are cleared through the CCP, it can seize the collateral of all defaulting bakers. Also, it receives the wheat that farmers sold forward. Hence, the CCP’s revenue consists of its total collateral holding \( kq_2 \) and the farmers’ wheat \( q_1 \). It uses a share \( 1 - \delta \) of this wheat to settle the forward contracts with bakers still alive, and each pays \( (p_{nf}^n - k)q_2 \) (and recall there is a measure \( 1 - \delta \) of bakers who do not default). The CCP sells the remainder \( \delta q_1 \) at the spot price. The CCP revenue – and, hence, its budget for paying farmers – is therefore given by

\[
R(\theta) = p(\theta) \delta q_1 + (p_{nf}^n - k)(1 - \delta)q_2 + kq_2
\]

and replacing the values for \( q_1 \) and \( q_2 \) and arranging, we obtain

\[
R(\theta) = p_{nf}^n + \delta (p(\theta) + k - p_{nf}^n).
\]

This is the amount of the asset all farmers get from the CCP.

Notice that bakers’ incentives are not affected by the CCP. Therefore, with novation, and given a collateral policy \( k \), the price \( p_{nf}^n(k) \) that leaves safe agents indifferent between participating in forward or spot trade is still given by (11) so that

\[
p_{nf}^n = 1 + (1 - \mu(\delta))k. \quad (14)
\]

Replacing this expression in (13), each farmer obtains a consumption level that equals

\[
(1 - \delta)(1 + (1 - \mu(\delta))k) + \delta (\theta + k) . \quad (15)
\]

Notice that, given a collateral \( k \), this amount is the expected consumption in state \( \theta \) without novation. So, novation acts as a substitute for diversification ex-ante. If farmers cannot perfectly diversify counterparty risk upfront – as we showed earlier – they can do so with a CCP that averages the exposure, and/or require a full guarantee against default. We now characterize the optimal collateral level, as the level that maximizes the farmers’ expected payoff.

**Proposition 3.** The optimal collateral policy with novation is \( k_n^* = 0 \).
Proof. The farmer’s payoff is given by

\[ U_1(k) = -q_1 + \int \log \left( p^n_f + \delta (p(\theta) + k - p^n_f) \right) dF(\theta). \]

The level of collateral that maximizes farmers ex-ante welfare solves \( \varphi_n(k) \equiv \partial U_1^k / \partial k \leq 0 \), or

\[ \varphi_n(k) = \int \frac{1 - (1 - \delta) \mu(\delta)}{(1 - \delta)(1 - \mu(\delta)k) + \delta \theta + k} dF(\theta) \leq 0. \]  

(16)

Since \( \mu(\delta) = \mu / (1 - \delta) \) and \( \mu > 1 \), we obtain \( \varphi_n(k) < 0 \), for all \( k \), so that the optimal collateral policy is \( k_n^* = 0 \). This completes the proof.

The intuition for the result is rather simple. We know that farmers’ consumption is decreasing in collateral. Since collateral is costly to produce, the forward price has to fall to leave bakers indifferent between trading forward or spot. However, with novation, the forward price is an element of the farmers’ revenue in all states, whether there is default or not. While collateral acted as a (costly) insurance device against default, this insurance is no longer needed with novation. Requiring collateral would just lower the revenue in all states, without providing more insurance. Hence, no collateral requirement is optimal. As a matter of fact, guaranteeing the price \( p_f^n \) is never optimal. Hence, novation is never equivalent to full insurance, where the forward contract is guaranteed.

We now compare the optimal collateral policy with and without novation. Clearly, given \( k_n^* = 0 \), we obtain that the optimal amount of collateral is lower with novation than without it, i.e. \( k^* > k_n^* \). Hence, novation helps to save collateral costs. This relatively modern risk management tool can be motivated along two different grounds. First, as pointed out earlier, it improves welfare through diversifying counterparty risk. But it also saves collateral costs in the optimal allocation. In other words, farmers benefit from novation not only when there is default, but even in normal times (although here there is always default).\(^{11}\)

Finally, the forward price under novation is higher than without it, \( p_i^n > p_i^k \). It is easy to check that farmers prefer novation with \( k_n^* = 0 \) to trading forward with collateral but no CCP, and so prefer novation to trading spot. Still, the CCP allocation with novation differs from the first best: The CCP’s revenue is a function of the aggregate state in period 2 and therefore it cannot fully insure farmer’s consumption. In the next section, we look at a policy that fully insures farmers against the aggregate shock.

\(^{11}\)With endogenous default, it is optimal to impose some collateral requirement. The reason is that collateral acts as an incentive device to lower default. The derivation is available upon request.
4.4.2 Novation And Mutualization

We now introduce *mutualization of losses*. As we show below, mutualization provides insurance to farmers, as it can eliminate the fluctuations in the CCP’s revenue. We first define mutualization and then analyze how mutualization affects the equilibrium.

When losses are mutualized, surviving bakers pay (or receive) an additional fee \( \phi(\theta) \) to the CCP.\(^{12}\) We denote by \( p^m_f \) the price of the forward contract with mutualization. Bakers who clear \( q_2 \) contracts with the CCP and who do not default therefore have to pay the CCP the amount \((p^m_f + \phi(\theta) - k) q_2\). As a consequence, in period 1, bakers solve

\[
\max_{q_2} -\mu k q_2 + (1 - \delta) \int V \left( (p(\theta) + k - p^m_f - \phi(\theta)) q_2 \right) dF(\theta)
\]

and the equilibrium forward price has to satisfy the no arbitrage condition

\[
p^m_f = 1 - \mu (\delta) k + \int \phi(\theta) dF(\theta).
\]

where we have used (4) to replace the expression for \( p(\theta) \). Notice that given \( k \) we have \( p^m_f < p^f = p^f_j \). The revenue of the CCP is given by

\[
R^m(\theta) = (1 - \delta) p^m_f + \delta (p(\theta) + k) + (1 - \delta) \phi(\theta).
\]

The CCP’s revenue is composed of three terms. The first two are derived from novating trades. The first term is the actual payment received from bakers that are still alive in period 2. The second term is the revenue from selling the wheat of deceased bakers on the spot market, while keeping their collateral. The net effect from the first two terms corresponds to the diversification of counterparty risk. Finally, the third term is the additional payment that alive bakers make to the CCP. Given \( k \), the revenue of the CCP is exactly the same as under no mutualization if \( \phi(\theta) = 0 \) for all \( \theta \).

Recall that farmers consume the CCP’s revenue and with no additional fee, the CCP’s revenue is state dependent. Hence, the CCP can provide insurance to farmers by varying \( \phi(\theta) \) to make the right-hand side of (19) constant. Given the efficient allocation is such that (1) farmers consume one unit of the asset and (2) no asset is produced in period 1, we now construct a policy \( \phi(\theta) \) such that (i) the CCP’s revenue is \( R^m(\theta) = 1 \) for all \( \theta \) and (ii) there is no collateral requirement, \( k = 0 \). Given such a payment schedule, the CCP can fully

\(^{12}\)If \( \phi(\theta) \) is negative, the fee is akin to a deductible for farmers in state \( \theta \), while in states where \( \phi(\theta) > 0 \), it is an insurance payment.
insure farmers – at the expected fair price of 1 for producing $q_1 = 1$ (farmer’s optimal choice of production) – while not relying on (inefficient) collateral to safeguard against default.\textsuperscript{13} This implies that for the given market structure the policy will implement the first best allocation.

Replacing (18) in (19) and using $R^m(\theta) = 1$, $k = 0$ and the expression for $p(\theta)$ in (4), we obtain that

$$\phi(\theta) = \frac{\delta (1 - \theta)}{1 - \delta} + \int \phi(\theta) dF(\theta)$$

where $\int \phi(\theta) dF(\theta) = C$ is an arbitrary constant. For $\theta < 1$, bakers pay more than the average, while they get a transfer whenever $\theta > 1$. Note that the fee schedule itself is indeterminate; it is defined only up to the constant $C$. The no-arbitrage condition ensures that the forward price adjusts fully to the average transfer between bakers and farmers through the payment schedule. Hence, bakers ex-ante expected utility is not affected by any payment schedule defined above. Farmers care only about the revenue of the CCP, but not about the forward price per se. Interestingly, setting the constant $C = 0$, we obtain that the forward price is again the expected spot price

$$p^m_f = 1 = \int p(\theta) dF(\theta).$$

Since there is no expected transfers between bakers and farmers, the forward price does not adjust at all and equals the (state uncontingent) CCP’s payment to a farmer for selling wheat forward. Here, farmers again care about the forward price that the CCP guarantees independent of the aggregate demand shock.

To conclude, novation and mutualization fully insure farmers against default and aggregate price risk. Most importantly, the exact mutualization scheme does not alter the incentives to trade in our economy, although it influences the terms of trade.\textsuperscript{14} In our environment, a CCP that novates forward contracts and mutualizes its losses attains the efficient allocation. There is no role left for regulatory policy. In the next section, we analyze the role of introducing a CCP on over-the-counter markets.

\textsuperscript{13}Indeed, if $k > 0$, one can never ensure a constant payment across states $\theta$ of at least 1. Integrating yields

$$\int R^m(\theta) dF(\theta) = 1 + k (1 - \mu) < 1.$$  

Hence, for some state the aggregate revenue must be less than 1.

\textsuperscript{14}In the Appendix, we show how to design the fee structure for any collateral level. Given this fee structure, we then show that it is optimal to impose no collateral requirement.
5 Over-The-Counter Market

We now modify our original environment to address, in a meaningful way, over-the-counter trading. To do so, we introduce a second type of wheat, special wheat. Each baker consumes only one type of special wheat. We use special wheat as a metaphor for those (generally exotic) contracts that are specially designed to fulfill the specific needs of the buyer.

5.1 The Model

The environment is essentially the same as before, except that now bakers have preferences defined over standard and special wheats. More precisely, baker $i$’s utility is $v_i(y, z, s) = (1 - \delta) [\sigma_i v(s) + \theta_i \log(y) - z] - \mu z$, where $v(0) = 0$ and $v(s)$ is concave. We assume $v'(0) = \infty$. The shock $\sigma_i \in [\underline{\sigma}, \bar{\sigma}]$ affects how much a baker likes his type of special wheat and it is drawn from a distribution $H$ in period 1. It is uncorrelated with the shock $\theta_i$.

We still denote the consumption of standard wheat by $y$, while $s$ denotes the special kind of wheat that only this baker consumes. In other words, no other baker derives utility from this special wheat. Bakers still need standard wheat, so that the spot market will still function.

Farmers can produce any type of wheat but only one type at a time. We will assume for the moment that farmers are as good as producing special or standard wheat. Hence, their utility is denoted by $U(q, x, s_i) = \log(x) - q - s_i$ where $s_i$ is the amount of special wheat they produce for baker $i$. Since farmers have to specialize, we have $s_is_j = 0$ for any $i \neq j$ and $qs_i = 0$ for any $i$. Finally, $s_i$ units of special wheat can be sold in the spot market for standard wheat, but at a discount $\lambda$. Therefore, if $s_i$ units of special wheat are liquidated, the seller can only get $\lambda p(\theta) s_i$ units of the asset. For the time being, we will assume $\lambda = 0$, so that special wheat is not fungible.

We first consider the problem of a planner who allocates the production of standard and special wheat. The planner is restricted by a technology that imposes that only one farmer can produce for a baker.\textsuperscript{15}

5.2 The Planner’s Problem

We restrict our attention to a planner that is only concerned by the allocation of special wheat, when there is a measure $1 - n$ of farmers producing special wheat and a measure $n$.

\textsuperscript{15}In general, the planner may have several farmers produce the same special type of wheat. For example, when $\sigma_i > 0$ for one baker only, then it is optimal that only his type of wheat is produced.
producing standard wheat. For the planner’s problem and what follows, it will be useful to specify the reservation utility of bakers as \( \bar{v} \) and the one of farmers as \( \bar{u} \). Given the number of farmers producing special wheat, the planner can only satisfy a measure \( 1 - n \) of bakers in special wheat. It is clear that the planner will want to serve all bakers with the highest preference shock \( \sigma \) only. Also, we already simplify the planner’s problem by disregarding the bakers’ production of the general good in the first period. This is without much loss of generality as such production is sub-optimal when \( \mu \) is large. Then, given \( n \), the planner’s problem is

\[
\max_{s_i \geq 0, x_i^1 \geq 0, x_i \geq 0} \int_{\tilde{\sigma}} (1 - \delta) \left[ \sigma_i v (s_i) - x_i^1 \right] + \log (x_i) - s_i dH (\sigma_i)
\]

subject to:

\[
(1 - \delta) \left[ \sigma_i v (s_i) - x_i^1 \right] \geq \bar{v}
\]

\[
\log (x_i) - s_i \geq \bar{u}
\]

\[
\int x_i dH (\sigma_i) = \int (1 - \delta) x_i^1 dH (\sigma_i)
\]

where \( H (\tilde{\sigma}) = n \). The planner faces two participation constraints, where \( \bar{u} \) and \( \bar{v} \) are the farmers’ and bakers’ reservation utility, as well as a resource constraint. The problem already takes into account the technological constraint: If \( s_i \) units of special wheat \( i \) is produced by a farmer then this is the consumption of baker \( i \). We will assume (without loss of generality) that \( \bar{v} = 0 \) and \( \bar{u} \) is finite. This implies that \( x_i > 0 \) for any \( i \) and farmers have to consume independently of whether they produce special wheat.

We solve this problem in details in the Appendix. Here, we want to concentrate on the economics. Clearly, the first order conditions of the unconstrained problem give us \( s_i^* \) and \( x_i^* \) such that \( (1 - \delta) \sigma_i v' (s_i^*) = 1 \) and \( x_i^* = 1 \), while the bakers’ production levels are indeterminate. The solution of the constrained problem is the following. When \( \sigma_i \) is very small (possibly zero when \( n = 0 \)), it is not worth to have the farmer produce for the baker. Since the baker does not receive any consumption, he will not produce either. When \( \sigma_i \) becomes larger, it is optimal for bakers to receive consumption \( s_i \). When \( s_i \) is relatively low, a farmer that consumes \( x^* \) can produce it without being constrained, i.e. \( -\bar{u} > s_i \). However, as \( \sigma_i \) increases some more, this constraint starts to bind. Therefore, there is some \( \sigma^* \) such that above \( \sigma^* \), the farmer’s participation constraint binds. The farmer is only willing to produce more for his baker if his consumption \( x_i \) is increased above \( x^* \). This introduces a distortion in the farmer’s consumption profile, so that it is not optimal to increase the baker’s con-

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16 In the Appendix, we consider the planner’s problem to allocate the production of special and standard wheat among farmers. This gives us the optimal size of each market.

17 In the implementation part, \( \bar{v} = 0 \), as a baker can only get special wheat from the OTC market, while \( \bar{u} \) will correspond to the farmer’s payoff from producing standard wheat.
umption to its first best level. Under relatively mild assumptions on preferences, we find that the bakers’s aggregate production is enough to compensate farmers.\footnote{In the Appendix, we also derive the solution when this condition is not satisfied. Then the main features of the solution are basically unchanged, except that the bakers’ participation constraint bind for all of them, so that \( x_i^2 = \sigma_i \cdot u(s_i) \).}

Figure XYZ shows the planner’s consumption allocations. There are several features to note. First, optimally farmers are completely insured against default risk. Second, below \( \sigma^* \), while farmers’ consumption is independent of who they meet (their consumption level does not depend on \( \sigma_i \)), their payoff is decreasing in \( \sigma_i \) but higher than the autarky level. Finally, above the threshold \( \sigma^* \), they are perfectly insured against who they meet, although they only get their reservation utility.

While we took the size of each market as given, it is clear that there is an efficient market size. In the Appendix, we solve for the complete planner’s problem, where the planner solve for the optimal size of each market. The qualitative properties of the solution are similar, and the optimal solution basically pins down the reservation payoff \( u \) in the planner’s problem.

### 5.3 The Equilibrium

We now seek a mechanism that can implement the constrained efficient allocation. We progressively introduce the features we are interested in to understand how they take care of existing frictions. First, we need to take a stance on the basic trading mechanism. Since we are interested in the effect of introducing a CCP on OTC market, it makes sense in our context to assume that the basic trading mechanism for special wheat is OTC trading. We make this notion more precise below.

Absent a forward market, the sequence of events is the following. First farmers decide whether they want to grow standard or special wheat. If they decide to grow standard wheat, they plant their seeds and move to period 2. If a farmer decides to grow special wheat, he is paired with a baker and we assume that the bargaining protocol is a take-it-or-leave-it offer by the farmer.\footnote{In the Appendix, we derive some results for the more general Nash bargaining.} If the offer is rejected (or they don’t find a match), they can still grow standard wheat. This is essentially a model of “one-sided” search, and this is how we model the OTC market.

It is useful to first consider the baker’s problem in period 2, when they enter the spot market with \( s_i \) units of special wheat, with \( w_i = p_i^o - k_i \) still to pay to the farmer (given the OTC
contract price is \( p_i^0 \) and they already pledged collateral \( k_i \). Their value function is then

\[
W(s_i, w_i) = \int \sigma_i v(s_i) - w_i + V(0) \, dG(\theta_i)
\]  

(22)

The decision in period 2 is still given by (2), and the envelope conditions give \( W_{s_i} = \sigma_i v'(s_i) \) and \( W_{w_i} = -1 \). We can now consider the problem of farmers making a take-it or leave-it offer \((s_i, k_i, w_i)\) in period 1,

\[
\max_{(s_i, k_i, w_i)} -s_i + (1 - \delta) \log (w_i + k_i) + \delta \log (k_i)
\]

s.t. \[\mu k_i + (1 - \delta) W(s_i, w_i) \geq (1 - \delta) W(0, 0)\]

The first order conditions give us \( p_i^0 = w_i + k_i \) is the price of an OTC contract

\[
v'(s_i) = v(s_i) \quad \text{(23)}
\]

\[
k_i = \delta p_i^0 / [\mu - (1 - \delta)] \quad \text{(24)}
\]

\[
p_i^0 = (1 - \delta) \sigma_i v(s_i) \quad \text{(25)}
\]

where the last equation is the participation constraint of bakers replacing for \( W(0, 0) \). Since the function \( v(\cdot) \) is type-independent, a feature of the solution is that all farmers want to produce the same quantity of special wheat independently of the type of bakers they meet.\(^\text{20}\)

We denote this quantity by \( \bar{s} \), defined as the solution to (23). Also, the offered price extracts all of the baker’s expected surplus. To complete the characterization of an equilibrium, notice that farmers must receive enough payoff from the OTC contract to participate in the OTC market, thus imposing

\[
-s_i + (1 - \delta) \log (p_i^0) + \delta \log (k_i) \geq -1 + \int \log \left( \frac{\theta}{n} \right) \, dF(\theta)
\]

where we have used market clearing in the spot market to determine the spot price. Replacing the terms of the OTC contract, we obtain

\[
1 - \bar{s} + \log ((1 - \delta) \sigma_i v(\bar{s})) + \delta \log \left( \frac{\delta}{\mu - (1 - \delta)} \right) \geq \int \log \left( \frac{\theta}{n} \right) \, dF(\theta)
\]

(26)

Given the number of farmers producing standard wheat, \( n \), (26) gives us a cut-off value \( \bar{s}(n) \), above which farmers prefer to produce special wheat for the baker. When \( n \) is large,

\(^\text{20}\)The offer equates the marginal benefit of producing more special wheat, which equals extracting an additional \( \sigma_i (1 - \delta) v'(s_i) \), to the (inverse) shadow value of the additional production, \( \sigma_i (1 - \delta) v(s_i) \).
the supply of standard wheat is large and the payoff from producing it is relatively small. Therefore, farmers have the incentives to produce special wheat, even though they do not get much for it (\( \tilde{\sigma} \) and \( p^0(\tilde{\sigma}) \) are relatively small).

Obviously, \( n \) is an equilibrium object and finding it boils down to a simple fix point problem. The number of bakers with a shock \( \sigma_i \) above the threshold \( \tilde{\sigma} \) in period 1 is \( 1 - H(\tilde{\sigma}(n)) / (1 - \delta) \). The probability that a farmer find an acceptable match is \( \min\{1 - H(\tilde{\sigma}(n)) / (1 - \delta), 1\} \), i.e. a match where the baker has \( \sigma_i > \tilde{\sigma} \). If a farmer does not find an acceptable match, then he can still sell standard wheat on the spot market. Therefore, the number of farmers producing standard wheat \( n \) is given by

\[
n = 1 - \min \left[ \frac{1 - H(\tilde{\sigma}(n))}{(1 - \delta)}, 1 \right]
\]

An OTC equilibrium is given by a threshold value \( \bar{\sigma} \) solving (26) with equality and a number of farmers \( n \) that satisfies (27). Notice that in equilibrium \( n > 0 \). When \( n = 0 \), there is no standard wheat, so the spot price in period 2 goes to infinity. This implies that no farmers would trade on the OTC market as \( \bar{\sigma} \) would tend to the upper bound of the distribution as well. We now derive a condition for the existence of an equilibrium where the OTC market is active.

**Proposition 4.** An OTC equilibrium exists with \( 0 < n < 1 \) if and only there is \( \sigma \) in the interior of the support of \( H \), such that

\[
1 - \bar{s} + \log((1 - \delta) \sigma v(\bar{s})) + \delta \log\left( \frac{\delta}{\mu - (1 - \delta)} \right) = \int \log(\theta) dF(\theta)
\]

**Proof.** Suppose there is such a \( \sigma \), then \( 1 - H(\sigma) > 0 \), and the probability of an acceptable match is positive. Since the return from these matches is positive, a farmer wants to search when he expects no other farmer to search. Since \( n = 0 \) is never an equilibrium, we must have in equilibrium \( n \in (0, 1) \).

Under the condition above, \( \bar{\sigma} \) solves

\[
1 - \bar{s} + \log((1 - \delta) \sigma_i v(\bar{s})) + \delta \log\left( \frac{\delta}{\mu - (1 - \delta)} \right) = \int \log\left( \frac{\theta(1 - \delta)}{(1 - \delta) + H(\bar{\sigma}) - 1} \right) dF(\theta)
\]

To summarize, farmers need to decide whether they want to specialize. If they do, they can extract all the surplus from bakers, although they face a default risk. If they don’t, they produce standard wheat, in which case they face competition from other farmers and an
aggregate valuation risk. Whenever they can extract a lot from bakers (their valuation of the special wheat is high), farmers prefer to bear the default risk and engage in the production of the special wheat. Otherwise, they will produce standard wheat. Since standard wheat has to be produced in equilibrium,\(^\text{21}\) the supply of special wheat is limited, in the sense that not every baker can consume special wheat.

The equilibrium allocation differs from the (constrained) first best allocation along several dimensions. First, farmers are not insured against the default risk. Second, our bargaining assumption implies that the redistribution of the payoff in equilibrium is the reverse of the optimal allocation: in equilibrium, farmers enjoy all the surplus from the production of special wheat, while bakers get none. As a consequence, farmers always get a higher payoff than is optimal, and are never insured against the preference shock \(\sigma_i\). Finally, the equilibrium allocation makes clear how one market affects the other.

We now turn to the same problem but when a forward contract is offered in the market for standard wheat and is CCP cleared. The result is obvious.

5.4 Forward And OTC Markets

We now introduce a forward market with an active CCP. We showed earlier that it was optimal that the CCP novates and mutualizes forward trades, and we will take this as given here. Since the utility is separable in the consumption of standard and special wheat, the optimal policy of the CCP is not modified by the presence of the OTC market. In particular, forward market trades are uncollateralized, \(k = 0\). On the OTC market, the fact that farmers can trade in the forward market affects their participation constraint. In particular, farmers will decide to trade OTC and make an offer to a baker of type \(\sigma_i\) if and only if

\[
1 - \bar{s} + \log \left( (1 - \delta) \sigma_i v(\bar{s}) \right) + \delta \log \left( \frac{\delta}{\mu - (1 - \delta)} \right) \geq \log \left( \frac{\int R(\theta) dF(\theta)}{n} \right) \tag{28}
\]

where \(R(\theta)/n\) is defined by (13). In particular, since only \(n\) farmers are active on the CCP market, the spot price is now \(p(\theta) = \theta/n\) and arbitrage gives us the forward price \(p_f = 1/n\). Replacing these expressions in \(R(\theta)\) we obtain that farmers make an offer if and only if

\[
1 - \bar{s} + \log \left( (1 - \delta) \sigma_i v(\bar{s}) \right) + \delta \log \left( \frac{\delta}{\mu - (1 - \delta)} \right) \geq \log \left( \frac{1}{n} \right) \tag{29}
\]

\(^{21}\)This follows from the Inada condition we imposed on the preference of bakers.
Equation (29) defines the threshold $\bar{\sigma}^f$ when it holds with equality. Let $n^f$ denote the equilibrium participation in the forward market, then the following result follows easily.

**Proposition 5.** $\bar{\sigma}^f > \bar{\sigma}$ and $n^f > n$, they are less deals on the OTC market when a CCP operates on the forward market.

### 5.5 Introducing CCP Clearing On The OTC Market

We now want to study the consequences of introducing a CCP on the OTC market. In particular, we want to study whether any CCP policy on the OTC market can get the economy closer to the planner’s (constrained) solution regarding the allocation of special wheat. We still retain our stark assumption that any kind of special wheat has no liquidation value. The CCP takes the OTC contract as given, but may require additional collateral. We first introduce novation and then mutualization.

#### 5.5.1 Novation

With novation, the CCP faces the problem that the special wheat has no liquidation value: When bakers default, the CCP cannot increase its revenue by selling the wheat on the spot market. We expect this limits the usefulness of the CCP. Let us suppose the CCP requires additional collateral $\tilde{k}_i$ whenever an OTC contract $(s_i, p_i^0, k_i)$ is submitted for clearance at the CCP.

The CCP’s revenue consists of its collateral holding $\int_{\bar{\sigma}} \left( k_i + \tilde{k}_i \right) dH(\sigma)$. The CCP also receives the promised amount of special wheat $s_i$ from each farmer. The amount of special wheat $s_i$ is then delivered to the baker $i$ if he is still alive. Each of them pays the CCP their respective dues $(p_i^0 - k_i - \tilde{k}_i)$.\(^{22}\) As the rest of the wheat has no liquidation value, the CCP just discards it. The CCP revenue is therefore given by

$$R^o = (1 - \delta) \int_{\bar{\sigma}} p_i^0 dH(\sigma_i) + \delta \int_{\bar{\sigma}} \left( k_i + \tilde{k}_i \right) dH(\sigma_i)$$  

\(^{(30)}\)

Note that, contrary to when a CCP operates on the forward market, $R^o$ is independent of the aggregate shock $\theta$ in the spot market in period 2. The CCP has to design a payment rule such that clearing through the CCP is incentive compatible for farmers. We did not have to

\(^{22}\)Notice that the CCP could consider only a share $\iota \in (0, 1)$ of the additional collateral requirement $\tilde{k}_i$ as a prepayment. We could then interpret $\iota \tilde{k}_i$ as a margin call and $(1 - \iota) \tilde{k}_i$ as a contribution to a default fund.
consider this constraint when we introduced a CCP for the forward market because all the contracts traded were homogenous and all farmers expected the same payoff. However, in the OTC market, farmers trade different contracts with different payoffs. If the CCP just split $R^o$ equally across all farmers, some farmers will get a lower utility than if they did not participate in CCP clearing.\footnote{Koeppl, Monnet and Temzelides (2010) also considers this incentive compatibility constraint.} Hence, when $m(s_i, p_i^o, k_i)$ denotes the CCP payment to a farmer with a contract $(s_i, p_i^o, k_i)$, we must require that the following participation constraint holds,

$$-s_i + \log (m(s_i, p_i^o, k_i)) \geq -s_i' + (1 - \delta) \log (p_i^o') + \delta \log (k_i')$$

(31)

where $(s_i', p_i^o', k_i')$ is the contract proposed when OTC traders do not clear through the CCP, as defined by (23)-(25).

Notice that we have implicitly required that $m(s_i, p_i^o, k_i)$ fully insures farmers against the risk of default. This does not have to be the case, but as farmers are risk averse, full insurance saves the CCP most resources. A payment schedule $\{m(s_i, p_i^o, k_i)\}_{s_i \geq \bar{s}}$ is feasible if

$$R^o \geq \int_{\bar{s}} m(s_i, p_i^o, k_i) dH(\sigma)$$

Many payment schedules are feasible and satisfy (31). The most natural payment schedule related to a contract $i$ is the expected revenue of the CCP for this contract,

$$m(s_i, p_i^o, k_i) = (1 - \delta) p_i^o + \delta (k_i + \tilde{k}_i).$$

(32)

When $\tilde{k}_i = 0$, the payment schedule is the certainty equivalent of contract $i$. In this sense, and as pointed out earlier, novation shares the default risk among farmers. Given this payment schedule, and using (22), the take-it or leave-it offer problem becomes

$$\max_{(s_i, p_i^o, k_i)} -s_i + \log \left((1 - \delta) p_i^o + \delta (k_i + \tilde{k}_i)\right)$$

s.t. $-\mu k_i - \mu \tilde{k}_i + (1 - \delta) \left(\sigma_i v(s_i) - (p_i^o - k_i - \tilde{k}_i)\right) \geq 0$

with first order conditions with respect to $s_i$, $p_i^o$ and $k_i$ respectively given by (and $\xi$ the
multiplier on the constraint)

\[-1 + \xi (1 - \delta) \sigma_i v' (s_i) = 0 \tag{33}\]

\[
\frac{1}{(1 - \delta) p_i^o + \delta (k_i + \tilde{k}_i)} \leq \xi
\]

\[
\frac{1}{(1 - \delta) p_i^o + \delta (k_i + \tilde{k}_i)} \leq \xi \frac{(\mu - 1 + \delta)}{\delta} \tag{35}\]

Since \(\mu - 1 + \delta > \delta\), the last constraint will not hold with an equality so that \(k_i = 0\).

Combining the first two conditions, we obtain

\[
(1 - \delta) p_i^o + \tilde{k}_i (\mu - 1) = (1 - \delta) \sigma_i v' (s_i) \tag{36}\]

Using the bakers’ participation constraint, we get

\[
(1 - \delta) \sigma_i (v (s_i) - v' (s_i)) = (\mu - 1) \tilde{k}_i \tag{37}\]

Equation (37) gives us the production of special wheat \(s_i\). It has the property that if the CCP provides additional insurance, through requiring additional collateral, \(s_i\) increases.\(^{24}\)

Finally notice that combining (32), (36) and \(k_i = 0\), we get

\[
m (s_i, p_i^o, k_i) = (1 - \delta) \sigma_i v' (s_i) + \tilde{k}_i \tag{38}\]

Given this payment schedule, it is easy to see that (31) is satisfied. Finally, we need to find the cut-off value \(\tilde{\sigma}^c\) such that offers are made for all \(\sigma_i \geq \tilde{\sigma}^c\). We suppose here that a CCP is also operating on the forward market and therefore \(\tilde{\sigma}^c\) is given by

\[
-s (\tilde{\sigma}^c) + \log ((1 - \delta) \tilde{\sigma}^c v (s (\tilde{\sigma}^c))) = -1 + \log \left(\frac{1}{n}\right) \tag{39}\]

where \(n\) is still given by (27).

Hence, given a collateral policy \(\tilde{k}_i\), an OTC and forward market equilibrium where a CCP clears trade on the OTC market and the forward market is given by \(n\) which solves,

\[
n = 1 - \min \left[\frac{1 - H (\tilde{\sigma}^c)}{(1 - \delta)}, 1\right],
\]

and where (31) holds for all farmers making an offer on the OTC market. The existence of

\(^{24}\)This follows from the concavity of \(v(.)\).
an equilibrium is then analogous to Proposition 4 provided that \( \tilde{k}_i \) is small enough for all \( i \).

We now characterize some properties of the equilibrium.

**Proposition 6.** If the payment scheme is given by (32), the optimal collateral policy is \( \tilde{k}_i = 0 \).

**Proof.** Notice that farmers’ consumption is given by \( p_i^o \) which is independent of \( \tilde{k}_i \). Also, \( \tilde{k}_i \) is payoff irrelevant for bakers, as they will require more special wheat \( s_i \) to reestablish their indifference between trading OTC or forward, as established by (37). However, producing more special wheat is costly for farmers, who do not enjoy more consumption. Therefore, farmers’ payoff is strictly decreasing in \( \tilde{k}_i \), while bakers’ payoff is independent of \( \tilde{k}_i \). This proves the claim.

This is a useful result, because then (37) gives us \( s_i = \tilde{s} \) for all \( i \) and (39) gives us simply

\[
\log ((1 - \delta) \bar{\sigma} v(\tilde{s})) = \tilde{s} - 1 + \log \left( \frac{1}{n} \right)
\]  

(40)

We also get the following result when we compare the equilibrium with and without CCP clearing on the OTC market,

**Proposition 7.** \( \sigma^f > \sigma^c \) and \( n^f > n^c \), i.e. there are more deals on the OTC market when a CCP clears OTC trades. Also \( p^c_f > p^f_f \), therefore farmers’ welfare is increased when a CCP clears OTC trades.

Since there are more farmers trading OTC, the price of standard wheat on the spot and forward markets increase. This benefits those farmers that do not find an appropriate match. Furthermore, on the OTC market, the CCP provides some insurance against default. and as a consequence, there are more deals, as the threshold valuation decreases. Therefore the welfare of farmers increases whether they participate on the OTC or on the forward market. Bakers are worse off as they have to pay more for standard wheat. We now study the mutualization of OTC trades.

### 5.6 Mutualization

As in the forward market, mutualization requires bakers with an OTC contract \( (s_i, p_i^o, k_i) \) who do not default to contribute an additional fee \( f_i \), where \( f_i \) can be either positive or
negative. Given the collateral policy \( \tilde{k}_i = 0 \), the CCP revenue with mutualization is then,

\[
R_m^o = (1 - \delta) \int_{\sigma_m} (p_i^o + f_i) dH(\sigma_i) + \delta \int_{\sigma_m} k_i dH(\sigma_i)
\]  

(41)

Notice that the CCP’s revenue is fixed and does not depend on the state in period 2. Therefore, there is no reason to make \( f_i \) state dependent in order to guarantee a given level of revenue. This implies that, given a policy \( \{f_i\} \), \( \int_{\sigma} f_i dH(\sigma_i) \) is constant.

We consider once again the payment schedule (32) where the CCP guarantees each farmer receives their expected payment, then the take-it or leave-it offer problem becomes:

\[
\begin{aligned}
\max_{(s_i, p_i^o, k_i)} & \quad -s_i + \log \left( (1 - \delta) p_i^o + \delta k_i \right) \\
\text{s.t.} & \quad -\mu k_i + (1 - \delta) (\sigma_i v(s_i) - p_i^o - f_i - k_i) \geq 0
\end{aligned}
\]

The first order conditions are still given by (33)-(35), so that \( k_i = 0 \). Also, \( p_i^o \) is given by (36). Using the bakers’ participation constraint, we get

\[
\sigma_i \left[ v(s_i) - v'(s_i) \right] = f_i
\]  

(42)

Given \( f_i \), this equation gives us \( s_i \) as an increasing function of \( f_i \).

To complete the characterization of the equilibrium with mutualization, we need to find the cut-off value \( \bar{\sigma}^m \) such that special wheat is produced for all \( \sigma_i \geq \bar{\sigma}^m \). We suppose here that another CCP is also operating on the forward market. Therefore, given a fee schedule \( f_i \), \( \bar{\sigma}^m \) is given by

\[
1 - s(f(\bar{\sigma}^m)) + \log \left( (1 - \delta) \bar{\sigma}^m v'(f(\bar{\sigma}^m)) \right) = \log \left( \frac{1}{n} \right)
\]  

(43)

---

25 It is easy to conduct the same analysis with first assuming a generic \( \tilde{k}_i \) and then showing that \( \tilde{k}_i = 0 \).

26 If \( f_i < 0 \), the production of farmers is reduced relative to the solution with no CCP as \( s_i < \bar{s} \). Also, as \( s_i \geq 0 \), we get that the fee is constrained from below by \( f_i \geq -\sigma_i (v(0) - v'(0)) \). Finally, given the payment schedule, the farmers participation constraint (31) adds a constraint on the magnitude of the transfer \( f_i \). Indeed, using (42) and given \( f_i \) we obtain the production level \( s_i(f_i) \). Replacing it in (31), and arranging, it then becomes,

\[
\begin{aligned}
-s_i + \log \left( m(s_i, p_i^o, k_i) \right) & \geq -s_i' + (1 - \delta) \log (p_i^o) + \delta \log (k_i'), \
\log (v'(s_i(f_i))) - s_i(f_i) & \geq \log (v(\bar{s})) + \delta \log \left( \frac{\delta}{\mu + \delta - 1} \right) - \bar{s}
\end{aligned}
\]

Clearly, setting \( f_i = 0 \) satisfies this condition since the OTC contract is the same contract as the no CCP clearing contract, but it also insures farmers against the default risk. This implies that farmers are also willing to accept a contract where \( f_i > 0 \), as long as \( f_i \) is not too large. It is easy to check that the left hand side is decreasing in \( f_i \) by concavity of the function \( v(\cdot) \).
where \( n \) is still given by (27). In particular, given \( n, \bar{s}m < \bar{s}c \) whenever \( f(\bar{s}m) < 0 \). There is therefore potential welfare gains from mutualization, as more OTC trades can take place. Then we obtain the following result.

**Proposition 8.** CCP-clearing on the OTC market implements the constrained optimal allocation with bargaining frictions.

The constrained optimal allocation with bargaining frictions is an allocation \((x'_i, s'_i)\) that solves the constrained planner’s problem, when we impose the additional restriction that bakers have no surplus. In the Appendix, we characterize this allocation. The main features of \((x'_i, s'_i)\) are that below a threshold \( \sigma' \), farmers’ consumption level is constant, while their production of special wheat increases with \( \sigma_i \). Also, above \( \sigma' \) their production of special wheat is constant at some level \( s' > \bar{s} \), and they are indifferent between producing special and standard wheat. In the Appendix, we show that the fee schedule \( f_i = \sigma_i [v(s'_i) - v'(s'_i)] \) implements the constrained allocation with bargaining. As a consequence, imposing our definition of mutualization, this is the best policy that a CCP can use. We show that the optimal fee schedule has the following property: \( f_i > 0 \) and increasing in \( \sigma_i \) for all \( \sigma_i > \sigma' \), so that the CCP increases the production of special wheat in the high valuation match, while it reduces the production of special wheat for bakers with low valuations.

Still a CCP that uses novation and mutualization does not achieve the constrained efficient allocation. The reason is that the CCP has to take the terms of the contracts as given. But they are set given a bargaining protocol and the frictions that arises from this bargaining protocol cannot be undone by the CCP restricted to use novation and mutualization. The CCP could circumvent this friction by negotiating the terms directly with bakers and farmers, but then the CCP would be engaged in the market well beyond the mere clearing and settlement of trades.\(^{27}\)

### 6 Discussion And Policy Implications

CCPs provide several services to centralized trading platforms. In the paper we have studied novation and the mutualization of risk, but CCPs also provide netting services and, while

\(^{27}\)For example, bakers would contact directly the CCP and the CCP would negotiate directly with bakers the terms for the production of special wheat. Then the CCP would negotiate with farmers the contractual terms of the production of the special wheat. It is clear that the CCP then would not face the matching frictions of the constrained planner: the CCP could require several farmers to produce the same type of wheat for one baker, which relaxes each farmer’s incentive constraint. At the same time, the CCP, knowing it’s aggregate resources, can provide the insurance to farmers’ against default risk. In addition, the CCP-dealer can make a payment to farmers that is independent of the type of special wheat they produce, or \( \sigma_i \).
they do not yet do it on a systematic basis, they could also provide aggregated information on trades.

Netting mainly offers cost savings in terms of posting collateral more efficiently.\textsuperscript{28} The provision of information on trades increases transparency, especially if the overall (i.e., market-wide) exposure of counterparties matter. The recent emergence of data warehouses speak to the benefit of providing information. Novation and mutualization \textit{directly} affect both the level of risk individual counterparties have to bear, and the overall allocation of risk in the market. This implies that these services have a direct influence on the incentives to trade and the incentives to take on risks.

6.1 CCP Services For Standard Contracts

Markets for standardized financial contracts are commonly organized around centralized trading platforms. Most of these markets clear trades centrally and over time have moved to rely on CCP services. Novation offers a cheap way to diversify counterparty risk.\textsuperscript{29} However, in the event of a default, the CCP has to buy/sell the asset that failed to be delivered/bought on the market and therefore assumes a price risk (a.k.a a replacement cost risk). This leaves some aggregate risk into trading. Mutualization distributes this risk among all members of the trading platform.

Our model can explain why combining novation and mutualization is efficient on a forward market. There, novation and mutualization offer perfect insurance against (1) counterparty default risk and (2) the price risk. Collateral could also play an insurance role, albeit a more expensive one, and is therefore sub-optimal in our simple model. In general, this is not the case, as collateral can still play a role as an incentive device and provide additional insurance against aggregate price risk. Nonetheless, the central message from our analysis is that novation and mutualization are all about reducing risk exposures. Our model is too simple to have a role for netting, but had we allowed for netting, we have no doubt that the essential role of netting would also have been to reduce risk.

\textsuperscript{28}See Duffie and XXX for an analysis of the degree of netting and contagion.

\textsuperscript{29}An argument can be made that novation is an essential technology that allows the introduction of electronic trading system where the counterparties are usually not directly observed.
6.2 Designing CCP Services For OTC Contracts

From our analysis, we conclude that there are important differences between operating a CCP on an OTC market or on a centralized one. OTC markets usually differ in their trading mechanisms and the types of contracts traded, i.e. their specificity and fungibility.

Even if contracts are not fungible, novation can still be a helpful tool to diversify counterparty risk on the OTC market. It is true that the more fungible the traded contracts are, the more scope there is to insure against an aggregate price risk. The extent to which such “traditional mutualization” increases efficiency depends on the degree of fungibility of the contracts traded. However, we showed that even if contracts are not fungible, such as on an OTC market, the CCP can go a long way in implementing the efficient allocation.

Contrary to novation that performs the same role for OTC as for standard contracts, we find that mutualization has a very different role on OTC markets. Because OTC contracts are generally quite specific and sometimes very exotic, it is difficult to give them a common value. In our model this is captured by our assumption that the valuation of the special wheat varies across buyers. For the same quantity of wheat traded (although not the same type), some buyers are willing to pay much more than others, and this is reflected in the different price of each contract \( p_i^o \). Therefore, while the default probability is the same across all buyers, the CCP’s replacement risk \( \delta p_i^o \) and its default exposure \( \delta s_i \) vary across contracts.

It is intuitive that given a permissible level of aggregate default exposure, it is optimal that those contracts with a high social value are allowed more risk than contracts with low social value. Hence, the CCP’s optimal default exposure should be skewed toward socially valuable contracts, by having more contracts with high valuation. Precisely, \( s_i (\sigma) \) should be increasing in \( \sigma \).

Absent mutualization, the frictions in our model are such that the size of contracts is independent of their valuation, i.e. \( s_i (\sigma) = \bar{s} \) for all \( \sigma \) where \( v' (\bar{s}) = v (\bar{s}) \). As argued above, this is suboptimal, and mutualization fixes this problem by inducing an increase in the size of contracts with high social value, while reducing the size of others. Indeed, we obtained that under mutualization the production is \( s (\sigma) = s' > \bar{s} \) for all \( \sigma > \sigma' \). Mutualization can do it by taxing the surplus from low valuation matches and subsidizing the one for high valuation matches. Hence, the optimal mutualization scheme achieves an efficient redistribution of default exposure given the terms of trades. However, those are influenced by the bargaining protocol used in OTC trades. Since the CCP takes the terms of the contract as given, mutualization comes short of achieving the efficient distribution of default exposures.
Now, turning to the replacement risk, we obtain that mutualization lowers it for high valuation contracts: Replacement cost risk is directly linked to the price of the contract, $\sigma_i v' (s_i)$. With novation, the size of each contract is $\bar{s}$, so that the price is $\sigma_i v' (\bar{s})$, while with mutualization, the size is $s' > \bar{s}$ for all valuation $\sigma > \sigma'$. Therefore, for those valuations, the price of the contract $- \sigma_i v' (s')$ will be lower under mutualization than novation. The intuition is simple: each contract is valued using the marginal benefit of special wheat. As more special wheat is produced, the marginal benefit declines, and so does the price of the contract, and therefore the replacement risk.

It is important to note that these conclusions hinge on our assumption that the CCP can condition its policy on the social value of each OTC contract it clears. However, it is easy to argue that this value is private information. Since mutualization requires a redistribution of payments across different types of transactions, a CCP is unlikely to implement such a service fully. However, the CCP could use some observable characteristics that correlate with the social value of the trades. For example, the identity of traders could be used, whenever fundamental traders carry trades with higher social value than, say, speculators. We leave the optimal design of a CCP policy under private information for future research.\footnote{See Koeppl, Monnet and Temzelides (2008) for an analysis of this issue.}

Finally, notice that a CCP does not make all farmers better off ex-post. Indeed, those who produce high valuation wheat would be better off without mutualization. Therefore, if some farmers could only produce high valuation wheat, they would object to the introduction of a CCP on the OTC market. In the same vein, it is not clear that bakers would favour the introduction of CCP for OTC trades. The reason is that by making OTC trades more attractive ex-ante for farmers, a CCP on the OTC market reduces the supply of standard wheat, and so increases its price. As bakers do not get any surplus on the OTC market, they are overall worse off. Still, the CCP achieves the efficient allocation.

### 6.3 The Importance of Dealers on OTC Markets

Dealers are an important infrastructure on OTC markets. Indeed, dealers might offer some type of diversification of counterparty risk. Hence, the need for a CCP offering traditional services might be greatly reduced especially considering the difficulties involved in running a mutualization scheme. Furthermore, dealers offer additional benefits as they tend to alleviate the trading (e.g., search) frictions inherent in OTC markets. This can partially explain why CCP solutions have not yet emerged on such markets.
What could then be a reason for introducing a CCP? OTC markets are by definition opaque and to a certain extent dominated by dealers that have market power due to this opaqueness. An information warehouse that makes terms of trades as well as overall positions public would improve transparency. This would lead to better assessments of risk and efficiency gains through more intense competition. Going beyond such a minimal solution, would also include some potential costs. One problem would be to concentrate risk in a single clearinghouse rather than spread across dealers. Furthermore, if dealers are not required to clear through this clearinghouse, there could be adverse selection increasing the risk within the CCP. Hence, future research needs to focus on the role of dealers when introducing a CCP on OTC markets.

7 Appendix

7.1 Proof of Proposition 2:

The payoff of type 1 is

\[ (1 - \delta) \log (p_f^k) + \delta \int \log (\theta + k) \, dF(\theta) \]

\[ = (1 - \delta) \log (1 - (\mu(\delta) - 1) k) + \delta \int \log (\theta + k) \, dF(\theta) \]

The level of collateral that maximizes the payoff of type 1 solves (where $U_k^1$ is the expected utility of type 1) $\varphi(k) \equiv \partial U_k^1 / \partial k = 0$, where

\[ \varphi(k) = \frac{1 - \mu(\delta)}{\theta_k + k} (1 - \delta) + \delta \int \frac{1}{\theta + k} dF(\theta) \]

(44)

and $\theta_k = 1 - \mu(\delta) k$. It is easy to check that the second order condition, $\varphi'(k) < 0$, is always satisfied for all $\delta$. Hence, it is optimal to require $k > 0$ unless $\varphi(0) \leq 0$. This is the case for the critical values of $\mu$ and $\delta$ where $\varphi(0) = 0$, or

\[ (1 - \mu(\delta)) (1 - \delta) + \delta \int \frac{1}{\theta} dF(\theta) = 0 \]

Solving for $\mu$ we obtain

\[ \mu(\delta) = 1 - \delta + \delta \int \frac{1}{\theta} dF(\theta) . \]
Since $\partial \varphi (0) / \partial \mu < 0$, $k^* > 0$, whenever $\mu < \mu (\delta)$ and $k^* = 0$ otherwise. Notice that $\mu (\delta) > 1$ for all $\delta > 0$ if and only if $\int \frac{1-\theta}{\delta} dF (\theta) > 1$ in which case $\mu (\delta) > 0$. To show the second part of the proof, use (11) so that $p_j^k = \bar{k}$ implies

$$
\bar{k} = \frac{1}{1 + \mu (\delta)}
$$

To show that $p_j^{k^*} > k^*$ we need to show that $\varphi (\bar{k}) < 0$, or using (105)

$$
1 - \delta + \int \frac{1}{\theta (1 + \mu (\delta)) + 1} dF (\theta) < \mu
$$

Since $\mu > 1$ and the integrand is less than one, the inequality always hold. This completes the proof.

### 7.2 Novation and Mutualization (Standard Wheat Only)

Given a collateral $k$, we can calculate the extra fee $\phi (\theta)$ in state $\theta$, as the difference between the novation payoff certainty-equivalent and the CCP revenue,

$$(1 - \delta) \phi (\theta) = \int R (\theta) dF (\theta) - R (\theta). \tag{45}$$

Notice in particular that the fee averages to zero, $\int \phi (\theta) dF (\theta) = 0$. As a consequence, for any given $k$, mutualization does not affect the forward market price,

$$p_f^m = 1 - \mu (\delta) k + k. \tag{46}$$

We now show that it is optimal to require no collateral.

**Proposition 9.** The optimal amount of collateral is $k_m^* = 0$.

**Proof.** With mutualization the optimal policy consists of choosing $k$ to maximize

$$\log \left( \int R (\theta) dF (\theta) \right) = \log \left[ (1 - \delta) \theta_k + k + \delta \int \theta dF (\theta) \right].$$

The first order condition (using $\theta_k = 1 - \mu (\delta) k$) is given by

$$\varphi_m (k) = \frac{1 - \mu}{(1 - \delta) \theta_k + k + \delta \int \theta dF (\theta)} < 0$$
so that the $k^*_m = 0$. 

Given farmers are insured against default with novation and mutualization, and collateral is costly, there is no need for collateral requirement. We now analyze the incentives of farmers to trade forward contracts vs. spot. With mutualization, their participation constraint is

$$\log \left( \int R(\theta) \, dF(\theta) \right) \geq \int \log(\theta) \, dF(\theta)$$

It follows directly from our analysis, that farmers prefer mutualization to only novation and always prefer mutualization to trading spot. Finally, replacing the expression for prices, quantities, etc. notice that $\int R(\theta) \, dF(\theta) = 1$. This is consumption level that farmers obtain absent any default risk. As a consequence, novation and mutualization perfectly insures farmers against the default risk. Incidentally, this shows that the policy function $\phi(\theta)$ above is optimal.

### 7.3 Optimal Allocation of Special Wheat

#### 7.3.1 The unconstrained planner’s problem

The unconstrained planner’s problem (scaled by the number of participants) is

$$\max_{s_i, x_i, x_2} \int \sigma \{ (1 - \delta) \left[ \sigma_i v'(s_i) - x_2^i \right] + \log(x_i) - s_i \} \, d\sigma_i$$

subject to

$$\int x_i d\sigma_i \leq \int (1 - \delta) x_2^i d\sigma_i \quad (\lambda^1)$$

$$x_2^i \geq 0 \quad (\lambda^2)$$

With first order condition

$$(1 - \delta) \sigma_i v'(s_i) = 1$$

$$1/x_i = \lambda^1$$

$$\lambda^1 = 1 - \lambda_2^i$$
Therefore, if $\lambda^i_2 = 0$ for all $i$ so that all bakers produce, then $\lambda^1 = 1$ and $x_i = 1$. The unconstrained efficient allocation is therefore given by
\[
(1 - \delta) \sigma_i v'(s_i) = 1 \\
x_i = 1.
\]

### 7.3.2 The constrained planner’s problem

The constrained planner’s problem takes care of the participation constraints:

\[
\max_{s_i, x^i_2, x^i_1} \int_{\sigma} \left\{ (1 - \delta) \left[ \sigma_i v(s_i) - x^i_2 \right] + \log (x_i) - s_i \right\} d\sigma_i \\
\text{s.t. } \int x_i dH(\sigma_i) = \int (1 - \delta) x^i_2 dH(\sigma_i) \quad (\lambda_1) \\
x^i_2 \geq 0 \quad (\lambda^i_2) \\
(1 - \delta) \left[ \sigma_i v(s_i) - x^i_2 \right] \geq \bar{v} \quad (\lambda^i_3) \\
\log (x_i) - s_i \geq \bar{u} \quad (\lambda^i_4)
\]

Where $\bar{v} = 0$ and $\bar{u}$ are the payoffs from the bakers’ and farmers’ outside options respectively. By the Inada condition, $s_i > 0$ and $x_i > 0$ for any $i$. The first order conditions are
\[
(1 - \delta) \sigma_i v'(s_i) = \frac{1 + \lambda^i_4}{1 + \lambda^i_3} \\
\frac{1}{x_i} = \frac{\lambda_1}{1 + \lambda^i_4} \\
\lambda_1 = 1 + \lambda^i_3 - \frac{\lambda^i_2}{1 - \delta}
\]

Since $s_i > 0$ and $\sigma_i > 0$ for all $i$, we can assume without loss of generality that $x^i_2 > 0$ for all $i$. Hence, $\lambda^i_2 = 0$. Therefore, we obtain a single first order necessary condition for all matches,
\[
(1 - \delta) \sigma_i v'(s_i) = x_i.
\]

Then the participation constraints give us the optimal allocation. First, notice that $\lambda^i_3 = \lambda_3$ for all $i$ as $\lambda^i_2 = 0$. Hence, we can consider two cases only.

**Case where $\lambda_3 = 0$.** In this case, we know $\lambda_1 = 1$. If $\lambda^i_4 = 0$, then the solution is the efficient allocation, i.e. $(x_i, s_i) = (x^*, s^*_i)$. Now, since $x_i = x^* = 1$ is constant, and $s^*_i$ is
increasing in \( \sigma_i \), the farmer’s PC may bing when \( \sigma_i \) is large, so that \( \lambda^i_4 > 0 \). In particular, this is the case for all \( \sigma_i > \sigma^* \) such that \( \log \left( (1 - \delta) \sigma^* v'(s^*_i) \right) = \bar{u} + s^*_i \). This is the case we study next.

If \( \lambda_3 = 0 \) and \( \lambda^i_4 > 0 \), the allocation for the match is

\[
\log \left( (1 - \delta) \sigma_i v'(s_i) \right) = \bar{u} + s_i \\
(1 - \delta) \sigma_i v'(s_i) = x_i > 1.
\]

Notice that farmers consume more than the efficient amount. In this way they can produce more for bakers. To recap, for \( \sigma_i < \sigma^* \), we get the efficient allocation and for \( \sigma_i > \sigma^* \), there is no surplus for farmers. Then we have to solve for \( x^i_2 \). Although the amount \( x^i_2 \) is indeterminate, we can solve for the aggregate amount necessary to sustain farmers’ consumption.

We already know that \( x_i = x^* \) for all \( \sigma_i < \sigma^* \) and \( x_i = x(\sigma_i) \) otherwise. Hence,

\[
\int_{\sigma} x^i_2 dH(\sigma_i) = \int_{\sigma} x^* + \int \sigma^* x(\sigma_i) dH(\sigma_i)
\]

A necessary and sufficient condition is that extracting all resources from bakers covers all the consumption from farmers, or

\[
\int_{\sigma} \sigma_i v(s_i) dH(\sigma_i) > \int_{\sigma} x^* + \int \sigma^* x(\sigma_i) dH(\sigma_i) \\
\int \frac{x(\sigma_i) v(s_i)}{v'(s_i)} dF(\sigma_i) > \int_{\sigma} x^* + \int \sigma^* x(\sigma_i) dH(\sigma_i)
\]

where \( x(\sigma_i) = x^* \) whenever \( \sigma_i < \bar{\sigma} \). This is sufficient, since the planner can then set \( x^i_2 = \sigma_i v(s_i) - \varepsilon \), for \( \varepsilon \) small enough. This is necessary, since the equilibrium we consider gives a positive surplus to all bakers, so that \( \sigma_i v(s_i) > x^i_2 \) for all \( i \). When this condition is not satisfied, the solution is given by the next case, where no baker has any surplus.

**Case where \( \lambda_3 > 0 \).** In this case, if \( \lambda^i_4 = 0 \), we get that

\[
x_i = \bar{x} = \frac{1}{1 + \lambda_3} < x^*.
\]

Hence, the payment is constant, but less than the efficient amount. The first order condition gives us \( s_i \) as the solution to

\[
(1 - \delta) \sigma_i v'(s_i) = \bar{x}.
\]
And since $\lambda_3 > 0$ we have $x_i^2 = \sigma_i v(s_i)$. To find $\bar{x}$, we will use the resource constraint, but we first need to solve for cases when $\lambda_4 > 0$. Indeed, in this case, where $x_i$ is constant, notice that $s_i$ is increasing in $\sigma_i$. Therefore, for $\sigma_i$ large, the farmer’s PC might bind.

If $\lambda_4 > 0$, we have $\log(x_i) - s_i = \bar{u}$. Given $\bar{x}$, there is $\bar{\sigma}$, such that for all $\sigma_i > \bar{\sigma}$, the farmer’s PC is violated if $s_i$ is set such that $(1 - \delta) \sigma_i v' (s_i) = \bar{x}$. Therefore, for all $\sigma_i > \bar{\sigma}$, we have $\lambda_4 > 0$ and the allocation is

$$
\log ((1 - \delta) \sigma_i v' (s_i)) = \bar{u} + s_i
$$

$$
(1 - \delta) \sigma_i v' (s_i) = x_i
$$

To complete the characterization of the constrained optimal solution when $\lambda_3 > 0$, we find $\bar{x}$ using the resource constraint,

$$
\int_{\bar{\sigma}}^{\hat{\sigma}} \bar{x} dH (\sigma_i) + \int_{\bar{\sigma}}^{\hat{\sigma}} x_i dH (\sigma_i) = \int_{\bar{\sigma}}^{\hat{\sigma}} \sigma_i v (\Phi (\bar{x})) dH (\sigma_i) + \int_{\bar{\sigma}}^{\hat{\sigma}} \sigma_i v (\Phi (x_i)) dH (\sigma_i)
$$

where $\Phi (x, \sigma) = v^{-1} (x / [(1 - \delta) \sigma])$.

**Summary of the constrained optimal allocation.** To summarize, we have two possible cases:

First, all bakers have some surplus. The allocation is then given by

$$(x_i, s_i) = \begin{cases} 
(x^*, s_i^*) & \text{for all } \sigma_i < \sigma^* \\
\log ((1 - \delta) \sigma_i v' (s_i)) = \bar{u} + s_i & \text{for all } \sigma_i \geq \sigma^*
\end{cases}
$$

Second, when the above allocation is not feasible, the optimal allocation is such that bakers have no surplus. The allocation is then given by

$$(x_i, s_i) = \begin{cases} 
\bar{x} < x^* & \text{for all } \sigma_i < \bar{\sigma} \\
(1 - \delta) \sigma_i v' (s_i) = \bar{x} & \text{for all } \sigma_i \geq \bar{\sigma}
\end{cases}
$$

and replacing $x_i^2 = \sigma_i v(s_i)$ in the resource constraint gives us $\bar{\sigma}$.
7.4 The Constrained Planner’s Problem With Bargaining Frictions

Note that \( k_i = 0 \) must be optimal as posting collateral reduces surplus. The take-it-or-leave-it friction implies that (i) the baker does not get any surplus, so that \( x_i^2 = \sigma_i v(s_i) \) and (ii) a match can always decide not to clear through the planner, but bilaterally. The planner’s problem is then given by

\[
\begin{align*}
\max_{(s_i, x_i)} & \int -s_i + \log(x_i) \, dH(\sigma) \\
\text{subject to} & \int x_i dH(\sigma) \leq (1 - \delta) \int \sigma_i v(s_i) dH(\sigma) \\
& -s_i + \log(x_i) \geq -\bar{s} + \log ((1 - \delta)\sigma_i v(\bar{s})) + \delta \log \left( \frac{\delta}{\mu - (1 - \delta)} \right)
\end{align*}
\]

where all the integrals are from \( \hat{\sigma} \), where \( H(\hat{\sigma}) = n \). The FOCs are given by

\[
\begin{align*}
x_i &= \frac{1 + \lambda_i}{\lambda} \\
(1 - \delta)\sigma_i v'(s_i) &= \frac{1 + \lambda_i}{\lambda}.
\end{align*}
\]

Replacing the first order condition in the PC and arranging, we obtain

\[
\bar{s} - s_i + \log \left( \frac{v'(s_i)}{v(\bar{s})} \right) \geq \delta \log \left( \frac{\delta}{\mu - (1 - \delta)} \right)
\]

so that \( s_i = s' \) is the same for all \( \sigma_i \) such that this constraint binds. Notice that \( s' > \bar{s} \) by concavity of \( v \). Indeed, the above equation holds with equality at \( s' \), and the right hand side is negative. At \( s_i = \bar{s} \), the LHS is zero and by concavity we need to increase \( s_i \) to reduce it so that the LHS equals the RHS.

Clearly, the solution has the same feature as in the case where there is no bargaining friction and there is no surplus for any bakers, i.e. the optimal allocation is \((x_i', s_i')\) where

\[
(x_i', s_i') = \begin{cases} 
  x_i' 
  & \text{for all } \sigma_i < \sigma' \\
  (1 - \delta) \sigma_i v'(s_i) = x_i' 
  & \text{for all } \sigma_i \geq \sigma'
\end{cases}
\]
and replacing \( x_i^2 = \sigma_i v'(s_i) \) in the resource constraint gives us \( x' \).

\[
\int_\sigma x' dH(\sigma_i) = \int_\sigma (1 - \delta) \sigma_i v'(\Phi(x_i)) dH(\sigma_i)
\]

where \( \Phi(x, \sigma) = v'^{-1}(x/[(1 - \delta) \sigma]) \) and where \( H(\delta) = 1 - n \). In particular, \( x' > 1 \) if the constrained efficient allocation with \( \lambda_3 = 0 \) is feasible and \( x' = \bar{x} < 1 \), otherwise. Also, notice that at the threshold \( \sigma' \), we have \( (1 - \delta) \sigma' v'(s') = x' \) so that \( x'_i > x' \) for all \( \sigma_i > \sigma' \).

### 7.5 Proof of Proposition 8:

**Proof.** The proof is by construction. We denote the optimal allocation with bargaining frictions by \((x', s')\), as characterized above. Recall that \((1 - \delta) \sigma_i v'(s'_i) = x'_i \). Set

\[
f_i = \sigma_i [v(s'_i) - v'(s'_i)]
\]

Since \( v \) is concave, the bargaining solution given \( f_i \) leads to the solution \( s_i = s'_i \). From (36), the price is then given by

\[
p_i^c = \sigma_i v'(s'_i) = \frac{x'_i}{1 - \delta}
\]

which given (32), induces a payment from the CCP to farmers of \( x'_i \).

Notice that by construction, the budget constraint of the CCP is fulfilled with equality and \( \int f_i dH(\sigma_i) = 0 \). Indeed, from (41), with \( k_i = 0 \) for all \( i \),

\[
\int_\sigma f_i dH(\sigma_i) = \int_\sigma \sigma_i [v(s'_i) - v'(s'_i)] dH(\sigma_i)
= \int_\sigma \sigma_i v(s'_i) dH(\sigma_i) - \int_\sigma \sigma_i v'(s'_i) dH(\sigma_i)
= \int_\sigma x'_i dH(\sigma_i) - \int_\sigma \frac{x'_i}{1 - \delta} dH(\sigma_i)
= 0.
\]

We now want to show that \( v(s') > v'(s') \) so that \( f_i > 0 \) for all \( \sigma_i > \sigma' \). Suppose this is not the case so that \( f_i < 0 \) for all \( \sigma_i \geq \sigma' \). Then for some \( \sigma_i < \sigma' \), we must have \( f_i(\sigma_i) > 0 \), since otherwise the above integral would not sum up to zero. Hence, for some \( \sigma_i < \sigma' \), \( \sigma_i v(s'_i) > \sigma_i v'(s'_i) = x' \), where the last inequality follows from the properties of the allocation. But we know that \( \sigma_i v(s'_i) \) is increasing in \( \sigma_i \) for all \( \sigma_i < \sigma' \). Hence, by continuity, it must be that \( \sigma_i v(s'_i) > \sigma_i v'(s'_i) \) also at \( \sigma_i = \sigma' \). This contradicts that \( f_i(\sigma') < 0 \).
Therefore we must have \( f_i(\sigma_i) < 0 \) for all \( \sigma_i < \sigma'' \) and \( f_i(\sigma_i) > 0 \) otherwise, where \( \sigma'' \) can be equal to \( \sigma' \). The important feature is that \( f(\sigma_i) > 0 \), for all \( \sigma_i \geq \sigma' \). 

### 7.6 Nash Bargaining

Suppose we use Nash bargaining instead of a take-it-or-leave-it offer, where \( \eta \) is the relative weight of farmers. Define the surplus of farmers and bakers as \( x_1 \) and \( x_2 \) respectively. Then

\[
x_2 = (1 - \delta) [\sigma_i v(s_i) - x^i_2]
\]
\[
x_1 = \log(x_i) - s_i - \bar{u}
\]

The first order conditions of the bargaining problem give

\[
p^o_1 = (1 - \delta) \sigma_i v'(s_i)
\]
\[
p^o_2 = \frac{\eta x_2}{(1 - \eta) x_1}
\]
\[
k_i = \frac{p^o_i}{\mu - (1 - \delta)}.
\]

Set \( \eta = 1/2 \) as in the social planner’s problem. This implies that the optimal \( s_i \) depends on \( \sigma \), as the second condition can be rewritten as

\[
\frac{v(s_i)}{v'(s_i)} - 1 = -s_i + \log((1 - \delta)\sigma v'(s_i)) + \delta \log\left(\frac{\delta}{\mu - (1 + \delta)}\right) - \bar{u}.
\]

Interestingly, the structure of the contract stays the same as the take-it-or-leave solution, the only difference being the amount \( s_i \) produced per match.

### 7.7 Nash Bargaining with Novation

With novation, the payment scheme we consider in the text implies that the bargaining solution gives

\[
p^o_i = \sigma_i v'(s_i)
\]

This yields the same condition as before except that farmers are fully insured against the default risk, so that

\[
\frac{v(s_i)}{v'(s_i)} - 1 = -s_i + \log((1 - \delta)\sigma v'(s_i)) - \bar{u}.
\]
Again, the structure of the contract remains the same relative to farmers get all surplus. But with novation, the amount of special wheat also changes. For each match there is less production of special wheat; i.e., $s_i$ declines across all $\sigma_i$. The solution is still different from the planner’s solution though: Both $s_i$ and $x_i$ vary here across all $\sigma$. And this is independent of the bargaining assumption.

7.8 The Constrained First Best Allocation

Each farmer is paired with a baker. If a baker is not matched with a farmer, the baker gets the planner allocation $(y(\theta_i), x'_1, x'_2(\theta_i))$, where we have set $s_i = 0$ (without loss of generality). Consider now a pair formed by one farmer and one baker with utility $\sigma_i$ for the special wheat. Each pair is parametrized by $\sigma_i$. The pair $\sigma_i$ can remain anonymous (i.e, unobservable to the planner) unless both members of the pair choose to be observed by the planner. If the pair remain anonymous, the baker and the farmer can bargain over the allocation. We assume that the farmer has all the bargaining power and makes a take-it or leave-it offer. Since farmers can produce either special or standard wheat, but not both, farmers either make a take-it or leave-it offer for standard wheat, or special wheat. If bakers accept, they can still claim to be single and get the singles allocation $(y(\theta_i), x'_1, x'_2(\theta_i))$ from the planner. If the farmer and the baker choose to be observed, the planner observes the true $\sigma_i$. Then the planner instructs farmers to produce $(q, s)(\sigma_i) = (q_i, s_i)$ where $q, s_i = 0$, since farmers can’t produce both type of wheat. Finally, the planner collects the production of general wheat and general good and redistribute it accordingly (all goods from observed match flow through the planner). For each pair $\sigma_i$, the planner’s allocation is then the farmer’s production of wheat $q(\sigma_i), s(\sigma_i) = (s_i, \sigma_i)$, the farmer’s consumption level in period 1 and 2 $(x_1(\sigma_i), x_2(\theta, \sigma_i)) = (x_{1,i}, x_{2,i}(\theta))$ in each state $\theta$, the baker’s production of the general good in period 1 and 2 $(x'_1(\sigma_i), x'_2(\theta, \varepsilon_i, \sigma_i)) = (x'_{1,i}, x'_{2,i}(\theta, \varepsilon_i))$ where $x'_{2,i}(\theta, \varepsilon_i)$ is his production of standard wheat in state $\theta$ when he gets a shock $\varepsilon_i$, and the baker’s wheat consumption basket $(y(\theta_i, \sigma_i), s(\sigma_i)) = (y_i(\theta, \varepsilon_i), s_i)$ where $y_i(\theta, \varepsilon_i)$ is his consumption of standard wheat in state $\theta$ when he gets a shock $\varepsilon_i$. We first consider the outcome of the bargaining game between a farmer and a baker.

7.8.1 Bargaining Outcomes

Bargaining Over Standard Wheat. If bakers and farmers bargain over the production of standard wheat, farmers make a take it or leave it offers which solves
\[
\max_{(c_i, k_i, w_i)} \quad -c_i + (1 - \delta) \int \log (w_i (\theta_i) + k_i) \, dF (\theta) \, dG (\varepsilon_i) + \delta \log (k_i)
\]

s.t. \quad -\mu k_i + (1 - \delta) \int (\theta_i \log (c_i) - w_i (\theta_i)) \, dF (\theta) \, dG (\varepsilon_i) \geq -\mu x'_1 + (1 - \delta) \int [\theta_i \log (y (\theta, \varepsilon_i)) - x'_2 (\theta, \varepsilon_i)] \, dF (\theta) \, dG (\varepsilon_i)

The first order conditions are

\[c_i = (1 - \delta) \lambda\]
\[(1 - \delta) \int \frac{1}{w_i (\theta_i) + k_i} \, dF (\theta) \, dG (\varepsilon_i) = (1 - \delta) \lambda\]
\[(1 - \delta) \int \frac{1}{w_i (\theta_i) + k_i} dF (\theta) dG (\varepsilon_i) + \frac{\delta}{k_i} = \lambda \mu\]

It is best for the farmer that \(w_i (\theta_i) = w_i\), in which case, we can simplify the FOC to

\[c_i = (1 - \delta) \lambda\]
\[w_i + k_i = \frac{(1 - \delta)}{c_i}\]
\[k_i = \frac{\delta}{c_i [\mu (\delta) - 1]}\]

The payoff of bakers is

\[\begin{align*}
-\mu \frac{\delta}{c_i [\mu (\delta) - 1]} + (1 - \delta) \log (c_i) - (1 - \delta) \left[ \frac{(1 - \delta)}{c_i} - \frac{\delta}{c_i [\mu (\delta) - 1]} \right] \\
= (1 - \delta) \log (c_i) - (1 - \delta) \left[ \frac{\mu - 1}{c_i [\mu (\delta) - 1]} \right] - \frac{\mu \delta}{c_i [\mu (\delta) - 1]}
\end{align*}\]

\[= (1 - \delta) \left[ \log (c_i) - \frac{1}{c_i} \right]\]

Therefore using the participation constraint of bakers in the bargaining problem, we obtain that \(c_i\) solves

\[\log (c_i) - \frac{1}{c_i} = -\mu (\delta) x'_1 + \int [\theta_i \log (y (\theta, \varepsilon_i)) - x'_2 (\theta, \varepsilon_i)] \, dF (\theta) \, dG (\varepsilon_i)\]

Since the solution \(c_i\) does not depend on \(\sigma_i\) we can write \(c_i = c\). Therefore, the payoff of the
farmer if he bargains is

\[-c + (1 - \delta) \log \left( \frac{1 - \delta}{c} \right) + \delta \log \left( \frac{\delta}{c \mu (\delta) - 1} \right)\]

**Bargaining Over Special Wheat.** If bakers and farmers bargain over the production of special wheat, farmers make a take-it or leave-it offer \((s_i, k_i, w_i)\) which solves

\[
\max_{(s_i, k_i, w_i)} -s_i + (1 - \delta) \log (w_i + k_i) + \delta \log (k_i)
\]

\[
\text{s.t.} \quad -\mu k_i + (1 - \delta) (\sigma_i s_i - w_i) \geq 0
\]

The first order conditions give us

\[
(1 - \delta) \sigma_i = w_i + k_i
\]

\[
k_i = \frac{\delta (1 - \delta) \sigma_i}{\mu - (1 - \delta)}
\]

\[
(1 - \delta) [s_i \sigma_i - w_i] = \mu k_i
\]

where the last equality follows from the fact that farmers extract all the surplus from bakers. Using the last equality we obtain \(s_i = 1\), so that the farmer’s payoff if he bargains is

\[-1 + (1 - \delta) \log ((1 - \delta) \sigma_i) + \delta \log \left( \frac{\sigma_i}{\mu (\delta) - 1} \right)\]

**7.8.2 The Planner’s Problem**

Given the bargaining solution and the planner’s allocation, agents in a pair will choose to be observed if and only if

\[
\int \log (x_{1,i} + x_{2,i} (\theta)) - q_i - s_i dF (\theta) \geq -c + (1 - \delta) \log \left( \frac{1 - \delta}{c} \right) + \delta \log \left( \frac{\delta}{c \mu (\delta) - 1} \right) \tag{58}
\]

\[
\int \log (x_{1,i} + x_{2,i} (\theta)) - q_i - s_i dF (\theta) \geq -1 + (1 - \delta) \log ((1 - \delta) \sigma_i) + \delta \log \left( \frac{\delta \sigma_i}{\mu (\delta) - 1} \right) \tag{59}
\]

\[
\int \sigma_i s_i + \theta_i \log (y_i (\theta, \varepsilon_i)) - x_{2,i} (\theta, \varepsilon_i) - \frac{\mu}{1 - \delta} x_{1,i} dFG (\theta_i) \geq \tag{60}
\]

\[
\int \theta_i \log (y (\theta, \varepsilon_i)) - x_{2,i} (\theta, \varepsilon_i) - \frac{\mu}{1 - \delta} x_{1,i} dFG (\theta_i)
\]

Hence, the planner is constrained by these three incentive compatibility (IC) constraints for each pair \(\sigma_i\). This IC set is complex as \(c\) is itself a function of the planner’s allocation. We
will now work toward simplifying the planner’s problem. But first, let us state the planner’s objective function. To write the objective function, notice that there are \( 1/(1 - \delta) \) bakers in period 1. Therefore the probability of a baker to meet a farmer is the farmer to baker ratio, or \( 1 - \delta \). Hence a measure 1 of bakers are matched, and \( \delta/(1 - \delta) \) bakers are unmatched. Finally, in period 2, a measure 1 of bakers that are matched with a farmer are still alive.

\[
\int \int \left[ \log (x_{1,i} (\theta)) - q_i - s_i \right] - \mu x'_{1,i} + (1 - \delta) \left[ \sigma_i s_i + \theta_i \log (y_i (\theta, \varepsilon_i)) - x'_{2,i} (\theta, \varepsilon_i) \right] dH (\sigma_i) dG (\varepsilon_i) dF (\theta) \\
+ \frac{\delta}{(1 - \delta)} \int -\mu x'_{1,i} + (1 - \delta) \left[ \theta_i \log (y (\theta, \varepsilon)) - x'_{2} (\theta, \varepsilon_i) \right] dG (\varepsilon_i) dF (\theta)
\]

First notice that it is optimal to have \( y (\theta, \varepsilon_i) > 0 \) and \( y_i (\theta, \varepsilon_i) > 0 \), as otherwise the utility of those bakers would be \(-\infty\). This implies that the planner will allocate the production of standard wheat to some farmers. Second, notice that for all \( \sigma_i < 1 \), the production of special wheat generates a negative surplus. Therefore, we can safely assume that \( s_i = 0 \) for all pair in which \( \sigma_i < \bar{\sigma} \). Finally, for those farmers who produce standard wheat, their consumption should not be depending on \( \sigma_i \), as this would introduce some harming uncertainty. Therefore, the planner will set \( x_{1,i} = x_1, \ x_{2,i} (\theta) = x_2 (\theta) \) and \( q_i = q \) for all pair in which \( q_i > 0 \) (the production is the same so as to minimize the production burden on each farmer, which relax all IC). Given this allocation, we now consider a new set of IC constraints, replacing (58) with instead

\[
\int \log (x_{1,i} + x_{2,i} (\theta)) - s_i dF (\theta) \geq \int \log (x_1 + x_2 (\theta)) - q dF (\theta) 
\]

In words, the planner’s allocation should give a higher payoff to farmers in a match \( \sigma_i \) who produce special wheat than the allocation of those farmers who produce the standard wheat. The idea is that those farmers who bargain over the production of standard wheat should be the ones who would anyway produce standard wheat in the planner’s allocation. Hence, for those farmers, (61) is not binding. Now, considering those farmers who should produce special wheat, if the planner’s allocation satisfies (61) then those farmers prefer the planner’s allocation to producing the standard wheat, so that they would not bargain over the production of standard wheat and (58) is satisfied. Therefore, we will solve the planner’s problem replacing (58) by (61) and then checking that the solution to the amended problem satisfies (58) for all farmers.

Finally, as \( \mu > 1 \) it is relatively easy to show that \( x_{1,i} = x_1 = x_{1,i}' = x_1' = 0 \).

Integrating all the considerations above into the (modified) planner’s problem, we maximize
the planner’s objective function with respect to \( (x, x', y, q) \):

\[
\int_{\theta} \int_{v} \left[ \log (x_{2,i} (\theta)) - s_i \right] + (1 - \delta) \left[ \sigma_i s_i + \theta_i \log (y_i (\theta, \varepsilon_i)) - x'_{2,i} (\theta, \varepsilon_i) \right] dH (\sigma_i) dG (\varepsilon_i) dF (\theta)
\]

\[
+ \int \int_{v} \log (x_2 (\theta)) - q + (1 - \delta) \left[ \theta_i \log (y_i (\theta, \varepsilon_i)) - x'_{2,i} (\theta, \varepsilon_i) \right] dH (\sigma_i) dG (\varepsilon_i) dF (\theta)
\]

\[
+ \delta \int \theta_i \log (y (\theta, \varepsilon_i)) - x'_2 (\theta, \varepsilon_i) dG (\varepsilon_i) dF (\theta)
\]

subject to:

\[
\lambda_q (\theta) : H (v) q \geq (1 - \delta) \int y_i (\theta, \varepsilon_i) dG (\varepsilon_i) H (\sigma_i)
\]

\[
+ \delta \int y(\theta, \varepsilon_i) dG (\varepsilon_i) , \text{ for all } \theta
\]

\[
\lambda_2 (\theta) : (1 - \delta) \int x'_{2,i} (\theta, \varepsilon_i) dG (\varepsilon_i) dH (\sigma_i) + \delta \int x'_2 (\theta, \varepsilon_i) dG (\varepsilon_i) \geq
\]

\[
\int_{\theta} x_{2,i} (\theta) dH (\sigma_i) + H (v) x_2 (\theta) , \text{ for all } \theta
\]

\[
\phi_{1,i} : \int \log (x_{2,i} (\theta)) - s_i dF (\theta) \geq \int \log (x_2 (\theta)) - q dF (\theta)
\]

\[
\phi_{2,i} : \int \log (x_{2,i} (\theta)) - s_i dF (\theta) \geq
\]

\[
-1 + (1 - \delta) \log ((1 - \delta) \sigma_i) + \delta \log \left( \frac{\sigma_i}{\mu (\delta) - 1} \right)
\]

\[
\phi_i : \int \sigma_i s_i + \theta_i \log (y_i (\theta, \varepsilon_i)) - x'_{2,i} (\theta, \varepsilon_i) dG (\varepsilon_i) dF (\theta)
\]

\[
\geq \int \theta_i \log (y (\theta, \varepsilon_i)) - x'_2 (\theta, \varepsilon_i) dG (\varepsilon_i) dF (\theta)
\]
The first order conditions give

\[ x_{2,i}(\theta) : \frac{1}{x_{2,i}(\theta)} - \lambda_2(\theta) + \frac{\phi_{1,i} + \phi_{2,i}}{x_{2,i}(\theta)} = 0 \]  
(67)

\[ x_2(\theta) : \frac{H(\bar{v})}{x_2(\theta)} - H(\bar{v})\lambda_2(\theta) - \int_0^1 \frac{\phi_{1,i}}{x_2(\theta)} dH(\sigma_i) = 0 \]  
(68)

\[ q : -H(\bar{v}) + H(\bar{v}) \int_0^1 \lambda_q(\theta) dF(\theta) + \int_0^1 \phi_{1,i} dH(\sigma_i) = 0 \]  
(69)

\[ s_i : (1 - \delta)\sigma_i - 1 - (\phi_{1,i} + \phi_{2,i}) + \sigma_i\phi_i \leq 0 \quad (= \text{ if } s_i > 0) \]  
(70)

\[ y_i(\theta, \varepsilon_i) : (1 - \delta) \frac{\theta_i}{y_i(\theta, \varepsilon_i)} - (1 - \delta)\lambda_q(\theta) + \phi_i \frac{\theta_i}{y_i(\theta, \varepsilon_i)} = 0 \]  
(71)

\[ y(\theta, \varepsilon_i) : \frac{\delta}{y(\theta, \varepsilon_i)} - \delta\lambda_q(\theta) - \int_0^1 \phi_i dH(\sigma_i) \frac{\theta_i}{y(\theta, \varepsilon_i)} = 0 \]  
(72)

\[ x_2(\theta, \varepsilon_i) : - (1 - \delta) + (1 - \delta)\lambda_2(\theta) - \phi_i \leq 0 \quad (= \text{ if } x_2(\theta, \varepsilon_i) > 0) \]  
(73)

\[ x_2'(\theta, \varepsilon_i) : - \delta + \delta\lambda_2(\theta) + \int_0^1 \phi_i dH(\sigma_i) \leq 0 \quad (= \text{ if } x_2'(\theta, \varepsilon_i) > 0) \]

To solve for the optimal allocation, we guess and then verify that \( \phi_i = 0 \) for all \( \sigma_i \). First consider the case where \( \phi_{1,i} + \phi_{2,i} = 0 \). Then (70) implies that this can only be the case for \( (1 - \delta)\sigma_i \leq 1 \) so that \( s_i = 0 \). Hence, \( s_i > 0 \) if and only if \( \phi_{1,i} + \phi_{2,i} > 0 \). Also, generically it must be that \( \phi_{1,i},\phi_{2,i} = 0 \) so that both IC constraint cannot be binding at the same time except for a measure zero of match.

Second, consider the case where \( s_i > 0 \). Then (70) together with \( \phi_i = 0 \) imply that

\[ (1 - \delta)\sigma_i - 1 = \phi_{1,i} + \phi_{2,i} \]

Therefore, replacing \( \phi_{1,i} + \phi_{2,i} \) in (67) we obtain

\[ \lambda_2(\theta) x_{2,i}(\theta) = (1 - \delta)\sigma_i \]  
(74)

Hence, if \( s_i > 0 \), then \( x_{2,i}(\theta) \) is an increasing function of \( \sigma_i \). Since \( \phi_i = 0 \), (71) and (72) imply that

\[ y_i(\theta, \varepsilon_i) = y(\theta, \varepsilon_i) \text{ for all } (\theta, \varepsilon_i) \]

Also, there must be production of the general good to obtain some consumption from farmers.
Hence, from (73) and/or (??) we obtain that for all $\theta$

$$\lambda_2 (\theta) = 1 \text{ for all } \theta$$

Using (68) and the fact that $\lambda_2 (\theta) = 1$, we obtain an expression for $\int_\theta \phi_{1,i} dH (\sigma_i)$, that we can replace in (69) to get

$$\int \lambda_q (\theta) dF (\theta) = x_2 (\theta) \equiv x_2$$

Hence, $x_2 (\theta)$ is independent of $\theta$. Using (74), $\lambda_2 (\theta) = 1$, and $\phi_i = 0$, we obtain that whenever $s_i > 0$ then

$$x_{2,i} (\theta) = (1 - \delta) \sigma_i.$$  \hspace{1cm} (75)

Now, consider the case where $\phi_{2,i} > 0$ and $s_i > 0$. Then replacing (75) in (65) we obtain an expression for $s_i$, as

$$s_i = 1 + \log ((1 - \delta) \sigma_i) - (1 - \delta) \log ((1 - \delta) \sigma_i) - \delta \log \left( \frac{\sigma_i}{\mu (\delta) - 1} \right)$$

$$s_i = 1 + \delta \log \left( 1 + \frac{\mu - 1}{\delta} \right)$$  \hspace{1cm} (76)

Notice that $s_i$ is independent of $\sigma_i$ whenever $\phi_{2,i} > 0$.

As $\sigma_i$ increases, notice that (64) first binds and then (65), as the right hand side of (65) is increasing in $\sigma_i$. Therefore there is $v$ such that $\phi_{1,i} > 0$ for all $v \in [\bar{v}, \tilde{v}]$ and $\phi_{1,i} = 0$ otherwise. Using (67) and (75) we end up with

$$\phi_{1,i} = (1 - \delta) \sigma_i - 1$$

so that integrating over all $v > \bar{v}$ we have

$$\int_\theta \phi_{1,i} dH (\sigma_i) = \int_\theta [(1 - \delta) \sigma_i - 1] dH (\sigma_i)$$

and replacing this expression in (68) together with $\lambda_2 (\theta) = 1$ we obtain

$$x_2 = 1 - \int_\theta [(1 - \delta) \sigma_i - 1] \frac{dH (\sigma_i)}{H (v)}$$  \hspace{1cm} (77)

Since $(1 - \delta) \bar{v} \geq 1$, it is easy to check that $x_2 (\theta) \leq 1$. Finally in the case in which $\phi_{1,i} > 0$ we have

$$s_i = \log ((1 - \delta) \sigma_i) - \log (x_2) + q$$  \hspace{1cm} (78)
Now, let us find the threshold $\bar{v}$ above which $\phi_{2,i} > 0$. This threshold is given by the value $\sigma_i$ such that (64) and (65) hold with equality, or

$$\log (x_2) - q = -1 + (1 - \delta) \log ((1 - \delta) \bar{v}) + \delta \log \left( \frac{\delta - \bar{v}}{\mu (\delta) - 1} \right).$$  \hfill (79)

Turning to the consumption of standard wheat of bakers, we can use (71) and (72) as well as the resource constraint (62) to obtain

$$\lambda_q (\theta) = \frac{\theta}{H (\bar{v}) q}$$

We can now use (68) and (69) to obtain $\int \lambda_q (\theta) dF (\theta) = x_2$, and replacing the expression for $\lambda_q (\theta)$ and using (77) we have

$$\frac{1}{q} = H (\bar{v}) - \int_{\bar{v}}^{\bar{v}_i} (1 - \delta) \sigma_i dH (\sigma_i)$$

And using (77) and (79) we obtain an expression for $\bar{v}$.\(^{31}\) Also,

$$y_i (\theta_i) = y_i (\bar{v}_i) = \frac{\theta_i}{H (\bar{v})} q$$  \hfill (80)

We now need $x'_{2,i} (\theta, \varepsilon_i)$ so that (66) is satisfied. Since $s_i \geq 0$, we get that (66) is naturally satisfied if $x'_{2,i} (\theta, \varepsilon_i) = \sigma_i + x'_{2} (\theta, \varepsilon_i)$ for all $\theta$.

Then (66) holds with strict inequality and $x'_{2} (\theta, \varepsilon_i)$ is given by the resource feasibility constraint (63):

$$(1 - \delta) \int_{\bar{v}}^{\bar{v}_i} v dH (v) + \int x'_{2} (\theta, \varepsilon_i) dG (\varepsilon_i) dF (\theta) = \int_{\bar{v}}^{\bar{v}_i} (1 - \delta) v dH (v) + H (\bar{v})$$

$$x'_{2} = \frac{H (v)}{H (\bar{v})}$$

Notice that linearity of payoff in $\sigma_i$ and $x'$ creates an indeterminacy for the planner. Indeed, the allocation $x'_{2,i} (\theta, \varepsilon_i) = s_i \sigma_i + x'_{2} (\theta, \varepsilon_i)$ is feasible and gives the same payoff to the planner. Therefore, the constrained first best is characterized by the allocation

\(^{31}\)This is the value solving

$$\log \left( 1 - \int_{\bar{v}}^{\bar{v}_i} [(1 - \delta) v_i - 1] \frac{dH (v_i)}{H (\bar{v})} \right) - \frac{1}{\gamma H (\bar{v}) - \gamma \int_{\bar{v}}^{\bar{v}_i} (1 - \delta) v_i dH (v_i)} = (1 - \delta) \log ((1 - \delta) \bar{v}) + \delta \log \left( \frac{\delta - \bar{v}}{\mu (\delta) - 1} \right)$$

Notice that $\bar{v} \neq \bar{v}$, so that there is no match where $\phi_{1,i} > 0$ is a possibility. Indeed, in this case, at $\bar{v} = 0$, the left hand side equals 1 while the right hand side equals $-\infty$. Also, at $\bar{v} = \infty$, the left hand side is negative, while the right hand side equals $\infty$. Since both sides are monotone in $\bar{v}$, there is a unique $\bar{v}$ such that the equality holds.
\((q (\sigma_i), y (\theta_i, \sigma_i), x' (\theta_i, \sigma_i), x_2 (\sigma_i), s (\sigma_i), \bar{v}, \tilde{v})\) such that

\[
x_2 = 1 - \int_{\bar{v}}^{\tilde{v}} [(1 - \delta) \sigma_i - 1] \frac{dH (\sigma_i)}{H (\bar{v})}
\]

\[
x_2 (\sigma_i) = (1 - \delta) \sigma_i
\]

\[
y (\theta_i) = y (\theta_i, \sigma_i) = \frac{\theta_i}{\bar{\theta}}
\]

\[
x'_2 = H (\bar{v})
\]

\[
x'_2 (\sigma_i) = \sigma_i + x'_2
\]

\[
q = q (\sigma_i) = \frac{1}{H (\bar{v})} \text{ for all } \sigma_i < \bar{v}
\]

\[
s (\sigma_i) = 1 + \delta \log \left(1 + \frac{\mu - 1}{\delta}\right) \text{ for all } \sigma_i \geq \bar{v}
\]

\[
s (\sigma_i) = \log \left(1 - \delta\right) \sigma_i - \log (x_2) + \frac{1}{H (\bar{v})} \text{ for all } \bar{v} \leq \sigma_i \leq \tilde{v}
\]

\[
\log \left(\frac{H (\bar{v})}{H (\bar{v}) (1 - \delta) \bar{v}} - \int_{\bar{v}}^{\tilde{v}} \frac{\sigma_i dH (\sigma_i)}{H (\bar{v})}\right) = \frac{1}{H (\bar{v})} - 1 - \delta \log \left(1 + \frac{\mu - 1}{\delta}\right)
\]

where the definition \(\bar{v}\) comes from the fact that the planner’s objective function is decreasing in \(\bar{v}\).

**Discussion.** Several aspects are worth noticing: first, the consumption of farmers who produce special wheat, \(x_2 (\sigma_i)\), is the expected valuation of the special wheat of their partner, \((1 - \delta) \sigma_i\). Hence, farmers are perfectly insured against the risk that “their” baker defaults. One might then expect that \(s = 1\), since then farmers extract the full expected surplus of bakers. However, the planner wants to have the production of special wheat as high as possible whenever \((1 - \delta) \sigma_i > 1\), as the marginal product is then always greater than the marginal cost. Therefore the production of special wheat is always such that one of the farmers’ constraints is binding. Notice that farmers produce more under the planner’s allocation than under the bargaining solution, as their payoff from consumption is also higher.

Second, what is the risk that farmers face? All agents with \(s_i = 0\) are perfectly insured against the risk aggregate risk \(\theta\). Also, farmers that produce the special wheat do not face any aggregate risk \(\theta\). However, these farmers face the valuation risk \(\sigma_i\). The planner insures farmers only partially against the valuation risk, because of the bargaining constraint (65). Absent this constraint, farmers would also be fully insured against the valuation risk (see
the first best problem below).

Finally, the farmers’ cost of producing the special wheat is born out only by the bakers who consume special wheat. However, the planner can change this somewhat, being wary that IC constraints have to be satisfied.

**Checking That (58) Is Satisfied.** To complete the characterization, we need to show that any pairs \( \sigma_i \) prefers the planner’s allocation to bargaining over the production of standard wheat. The constraint (64) ensures that farmers prefer to produce the special wheat when the planners instruct him to do so. When a farmer has to produce standard wheat, his payoff is

\[
\log (x(\theta)) - q = \log ((1 - \delta \bar{v}) - 1 - \delta \log \left(1 + \frac{\mu - 1}{\delta}\right) \tag{81}
\]

While if the farmer bargains, his payoff is

\[-c + (1 - \delta) \log \left(\frac{1 - \delta}{c}\right) + \delta \log \left(\frac{\delta}{c(\mu(\delta) - 1)\right}\]

where \( c \) is given by the baker’s participation constraint

\[
\log (c) - \frac{1}{c} = \int \theta_i \log \left(\frac{\theta_i}{\bar{\theta}}\right) dF(\theta) dG(\varepsilon_i) - H(\bar{v})
\]

\[
\log (c) = \frac{1}{c} + \int \theta_i \log \left(\frac{\theta_i}{\bar{\theta}}\right) dF(\theta) dG(\varepsilon_i) - H(\bar{v}) \tag{82}
\]

So that the planner’s allocation is IC if and only if

\[
\log ((1 - \delta \bar{v}) - 1 - \delta \log \left(1 + \frac{\mu - 1}{\delta}\right) \geq -c + \log \left(\frac{1 - \delta}{c}\right) - \delta \log \left(1 + \frac{\mu - 1}{\delta}\right)
\]

\[
\log ((1 - \delta \bar{v}) - 1 \geq -c + \log \left(\frac{1 - \delta}{c}\right)
\]

\[
\log ((1 - \delta \bar{v}) - 1 \geq -c - \log (c) + \log (1 - \delta)
\]

And replacing the expression for \( \log (c) \) using (82) the necessary and sufficient condition becomes

\[
\log ((1 - \delta \bar{v}) \geq 1 + \log (1 - \delta) - c - \frac{1}{c} - \int \theta_i \log \left(\frac{\theta_i}{\bar{\theta}}\right) dG(\varepsilon_i) dF(\theta) + H(\bar{v})
\]
Notice that the right hand side is maximized at \( c = 1 \). Therefore, replacing \( c = 1 \) we obtain that a sufficient condition for IC is

\[
\log((1 - \delta) \bar{v}) \geq H(\bar{v}) - 1 + \log(1 - \delta) - \int \theta_i \log \left( \frac{\theta_i}{\bar{\theta}} \right) dG(\varepsilon_i) dF(\theta)
\]

It is easy to check that the function \( (\theta + \varepsilon_i) \log \left( (\theta + \varepsilon_i) / \theta \right) \) is convex in \( \varepsilon_i \). Therefore, Jensen inequality implies that

\[
\int \theta_i \log \left( \frac{\theta_i}{\bar{\theta}} \right) dF(\theta) dG(\varepsilon_i) \geq 0
\]

Since \( (1 - \delta) \bar{v} > 1 \), the left hand side is positive, so that the inequality is always satisfied. This implies that farmers will always prefer the planner’s allocation to making a take-it or leave offer to bakers.

### 7.9 The first best allocation

We now solve the planner’s allocation when the planner only faces a participation constraint on farmers. More precisely, while the planner cannot observe whether a baker is matched with a farmer and the specific match type of a farmer, farmers and bankers cannot bargain. In this case, the planner does not face IC constraints (58)-(60). However the allocation still has to satisfies (61), so that farmers have to enjoy at least the utility of producing the standard wheat. Integrating the fact that below the cutoff value \( \hat{v} \), the planner will set \( s_i = 0 \), we have the following problem:

\[
\begin{align*}
\int_{\hat{v}} \int [\log(x_{2,i}(\theta)) - s_i] + (1 - \delta) [\sigma_i s_i + \theta_i \log(y_i(\theta, \varepsilon_i)) - x'_{s,i}(\theta, \varepsilon_i)] dH(\sigma_i) dG(\varepsilon_i) dF(\theta) \\
+ \int \int_{\hat{v}} \log(x_2(\theta)) - q dH(\sigma_i) dF(\theta) \\
+ (1 - (1 - \delta)(1 - H(\hat{v}))) \int \theta_i \log(y(\theta, \varepsilon_i)) - x'_2(\theta, \varepsilon_i) dG(\varepsilon_i) dF(\theta)
\end{align*}
\]

subject to: \( q_i s_i = 0 \) and
\[ \lambda_q(\theta) : H(\hat{v})q \geq (1 - \delta) \int_{\hat{v}} y_i(\theta, \varepsilon_i) dG(\theta_i) H(\sigma_i) + (1 - (1 - \delta) (1 - H(\hat{v}))) \int y(\theta, \varepsilon_i) dG(\varepsilon_i), \text{ for all } \theta \]

\[ \lambda_2(\theta) : (1 - \delta) \int_{\hat{v}} x'_{2,i}(\theta, \varepsilon_i) dG(\varepsilon_i) dH(\sigma_1) + (1 - (1 - \delta) (1 - H(\hat{v}))) \int x'_{2,i}(\theta, \varepsilon_i) dG(\varepsilon_i), \text{ for all } \theta \]

\[ \phi_{1,i} : \int \log(x_{2,i}(\theta)) dF(\theta) - s_i = \int \log(x_2(\theta)) dF(\theta) - q \quad (85) \]

The planner’s first order conditions are with the understanding that \( q_i = q \) whenever \( s_i > 0 \),

\[ x_{2,i}(\theta) : \frac{1}{x_{2,i}(\theta)} - \lambda_2(\theta) + \frac{\phi_{1,i}}{x_{2,i}(\theta)} = 0 \quad (86) \]

\[ x_2(\theta) : \frac{H(\hat{v})}{x_2(\theta)} - H(\hat{v}) \lambda_2(\theta) - \int \frac{\phi_{1,i}}{x_2(\theta)} dH(\sigma_i) = 0 \quad (87) \]

\[ q : -H(\hat{v}) + H(\hat{v}) \int \lambda_q(\theta) dF(\theta) + \int \phi_{1,i} dH(\sigma_i) = 0 \quad (88) \]

\[ s_i : ((1 - \delta) \sigma_i - 1) - \phi_{1,i} \leq 0 \quad (= \text{ if } s_i > 0) \quad (89) \]

\[ y_i(\theta, \varepsilon_i) : \frac{\theta_i}{y_i(\theta, \varepsilon_i)} - \lambda_q(\theta) = 0 \quad (90) \]

\[ y(\theta, \varepsilon_i) : \frac{\theta_i}{y(\theta, \varepsilon_i)} - \lambda_q(\theta) = 0 \quad (91) \]

\[ x'_{2,i}(\theta, \varepsilon_i) : -1 + \lambda_2(\theta) \leq 0 \quad (= \text{ if } x'_{2,i}(\theta_i) > 0) \quad (92) \]

\[ x'_{2,i}(\theta, \varepsilon_i) : -1 + \lambda_2(\theta) \leq 0 \quad (= \text{ if } x'_{2,i}(\theta_i) > 0) \quad (93) \]

All farmers will enjoy the payoff of producing the standard wheat, independently of whether they actually produce standard or special wheat. The reason is that if \( \phi_{1,i} > 0 \) then \( s_i > 0 \) so that farmers produce special wheat, nonetheless as (61) is binding, they obtain the payoff of standard wheat producers,

\[ \int \log(x_{2,i}(\theta)) dF(\theta) - s_i = \int \log(x_2(\theta)) dF(\theta) - q. \quad (94) \]

Since there is some production of general good, (93) implies that \( \lambda_2(\theta) = 1 \).

Now consider the case where \( s_i > 0 \). Then (89) implies that for all \( v > \hat{v} \),

\[ \phi_{1,i} = ((1 - \delta) \sigma_i - 1) \quad (95) \]
Therefore, replacing $\phi_{1,i}$ in (86) and $\lambda_2(\theta) = 1$, we obtain

$$x_{2,i}(\theta) = (1 - \delta)\sigma_i$$

(96)

Hence, if $s_i > 0$, then $x_{2,i}(\theta)$ is an increasing function of $\sigma_i$.

Since $\phi_{1,i} = 0$ whenever $s_i = 0$, that is whenever $v \leq \hat{v}$, we obtain using (95)

$$\int \phi_{1,i} dH(\sigma_i) = \int_{\hat{v}} [(1 - \delta)\sigma_i - 1] dH(\sigma_i)$$

and plugging this expression in (87) and using $\lambda_2(\theta) = 1$, we get

$$x_2(\theta) = \frac{1}{H(\hat{v})} - \int_{\hat{v}} (1 - \delta)\sigma_i dH(\sigma_i)$$

(97)

Also, using (87) and (88) we obtain

$$x_2(\theta) = \int \lambda_q(\theta) dF(\theta)$$

(98)

However, using the resource constraint (83) together with (90) and (91) we obtain

$$\lambda_q(\theta) = \frac{\theta}{H(\hat{v})q}$$

so that

$$y(\theta, \varepsilon_i) = y_i(\theta, \varepsilon_i) = \frac{\theta_i}{\theta} H(\hat{v}) q.$$  

(99)

We can then find $q$ by using (97) and (98) as

$$\frac{1}{q} = 1 - \int_{\hat{v}} (1 - \delta)\sigma_i dH(\sigma_i)$$

(100)

We obtain $s_i$ from the binding IC constraint (85),

$$s_i = \log ((1 - \delta)\sigma_i) + q - \log \left( \frac{1}{H(\hat{v})q} \right).$$

(101)

Also, $\hat{v}$ is such that $(1 - \delta)\hat{v} = 1$ and all $\sigma_i > \hat{v}$ are such that the $\phi_{1,i} > 0$. Finally, from
the resource constraint (84) we obtain

\[
(1 - \delta) \int \int x'_{2,i} (\theta, \varepsilon_i) dG (\varepsilon_i) dH (\sigma_i) + (1 - (1 - \delta) (1 - H (\hat{v}))) \int x'_{2} (\theta, \varepsilon_i) dG (\varepsilon_i)
\]

\[
= \int \int (1 - \delta) \sigma_i dH (\sigma_i) + H (\hat{v}) x_2 (\theta)
\]

(102)

and setting \( x'_{2,i} (\theta, \varepsilon_i) = \sigma_i + x'_{2} (\theta, \varepsilon_i) \), we get

\[
(1 - \delta) \int \sigma_i dH (\sigma_i) + \int x'_{2} (\theta, \varepsilon_i) dG (\varepsilon_i) = \int (1 - \delta) \sigma_i dH (\sigma_i) + H (\hat{v}) x_2 (\theta)
\]

\[
x'_{2} (\theta) = H (\hat{v}) x_2 (\theta)
\]

(103)

\[
x'_{2} (\theta) = \frac{1}{q}
\]

(104)

The allocation is then

\[
q = \frac{1}{1 - \int \sigma_i dH (\sigma_i)}
\]

\[
x_2 (\theta) = \frac{1}{H (\hat{v}) q}
\]

\[
x'_{2} (\theta) = \frac{1}{q}
\]

\[
x'_{2,i} (\theta) = \sigma_i + x'_{2} (\theta)
\]

\[
x_{2,i} (\theta) = (1 - \delta) \sigma_i
\]

\[
s_i = \log ((1 - \delta) \sigma_i) + q - \log \left( \frac{1}{H (\hat{v}) q} \right)
\]

\[
y (\theta, \varepsilon_i) = y_i (\theta, \varepsilon_i) = \theta_i H (\hat{v}) q.
\]

Given this allocation the planner’s objective function is:

\[
\int \int [\log (x_{2,i} (\theta)) - s_i] + (1 - \delta) \left[ \sigma_i s_i + \theta_i \log (y_i (\theta, \varepsilon_i)) - x'_{2,i} (\theta, \varepsilon_i) \right] dH (\sigma_i) dG (\varepsilon_i) dF (\theta)
\]

\[
+ \int \int \log (x_2 (\theta)) - q dH (\sigma_i) dF (\theta)
\]

\[
+ (1 - (1 - \delta) (1 - H (\hat{v}))) \int \theta_i \log (y (\theta, \varepsilon_i)) - x'_{2} (\theta, \varepsilon_i) dG (\varepsilon_i) dF (\theta)
\]

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\[
\int_{\hat{v}} \int (1 - \delta) \sigma_i (s_i - 1) dH (\sigma_i) dG (\varepsilon_i) dF (\theta) + \log (x_2) - q \\
+ \int \theta_i \log (y (\theta, \varepsilon_i)) dG (\varepsilon_i) dF (\theta) - x'_2
\]

\[
\int_{\hat{v}} (1 - \delta) \sigma_i \left( \log ((1 - \delta) \sigma_i) + q - \log \left( \frac{1}{H (\hat{v}) q} \right) - 1 \right) dH (\sigma_i) \\
+ \log \left( \frac{1}{H (\hat{v}) q} \right) - q \\
+ \int \theta_i \log \left( \frac{\theta_i}{\hat{\theta}} H (\hat{v}) q \right) dG (\varepsilon_i) dF (\theta) - \frac{1}{q}
\]

\[
\int_{\hat{v}} (1 - \delta) \sigma_i [\log ((1 - \delta) \sigma_i) - 1] dH (\sigma_i) \\
+ [- \log (H (\hat{v}) q) - q] \left[ 1 - \int_{\hat{v}} (1 - \delta) \sigma_i dH (\sigma_i) \right] \\
+ \int \theta_i \log \left( \frac{\theta_i}{\hat{\theta}} \right) dG (\varepsilon_i) dF (\theta) - \frac{1}{q} + \log (H (\hat{v}) q)
\]

using
\[
1 - \int_{\hat{v}} (1 - \delta) \sigma_i dH (\sigma_i) = \frac{1}{q}
\]

we get
\[
\int_{\hat{v}} (1 - \delta) \sigma_i [\log ((1 - \delta) \sigma_i) - 1] dH (\sigma_i) \\
+ \int \theta_i \log \left( \frac{\theta_i}{\hat{\theta}} \right) dG (\varepsilon_i) dF (\theta) - \frac{1}{q} - 1 + \log (H (\hat{v}) q) \left( 1 - \frac{1}{q} \right)
\]

\[
\int_{\hat{v}} (1 - \delta) \sigma_i \log ((1 - \delta) \sigma_i) dH (\sigma_i) - 1 + \frac{1}{q} \\
+ \int \theta_i \log \left( \frac{\theta_i}{\hat{\theta}} \right) dG (\varepsilon_i) dF (\theta) - \frac{1}{q} - 1 + \log (H (\hat{v}) q) \left( 1 - \frac{1}{q} \right)
\]

\[
\int_{\hat{v}} (1 - \delta) \sigma_i \log ((1 - \delta) \sigma_i) dH (\sigma_i) - 1 \\
+ \int \theta_i \log \left( \frac{\theta_i}{\hat{\theta}} \right) dG (\varepsilon_i) dF (\theta) - 1 + \log \left( \frac{1}{H (\hat{v}) q} \right) \left( \frac{1}{q} - 1 \right)
\]
replacing \( q \):

\[
\int_{\hat{\theta}} (1 - \delta) \sigma_i \log ((1 - \delta) \sigma_i) dH(\sigma_i) - 1
\]

\[+ \int \theta_i \log \left( \frac{\theta_i}{\theta} \right) dG(\varepsilon_i) dF(\theta) - 1 - \log \left( \frac{1 - \int_{\hat{\theta}} (1 - \delta) \sigma_i dH(\sigma_i)}{H(\hat{\theta})} \right) \int_{\hat{\theta}} (1 - \delta) \sigma_i dH(\sigma_i)
\]

The FOC with respect to \( \hat{\theta} \):

\[
-(1 - \delta) \hat{\theta} \log ((1 - \delta) \hat{\theta}) h(\hat{\theta}) + \log \left( \frac{1}{H(\hat{\theta})} \right) (1 - \delta) \hat{\theta} h(\hat{\theta})
\]

\[+ qH(\hat{\theta}) \left( \frac{1}{q} - 1 \right) \left[ \frac{(1 - \delta) \hat{\theta} h(\hat{\theta}) H(\hat{\theta}) - h(\hat{\theta}) + h(\hat{\theta}) \int_{\hat{\theta}} (1 - \delta) \sigma_i dH(\sigma_i)}{H(\hat{\theta})^2} \right]
\]

\[
-(1 - \delta) \hat{\theta} \log ((1 - \delta) \hat{\theta}) h(\hat{\theta}) + \log \left( \frac{1}{H(\hat{\theta})} \right) (1 - \delta) \hat{\theta} h(\hat{\theta})
\]

\[+ qH(\hat{\theta}) \left( \frac{1}{q} - 1 \right) h(\hat{\theta}) \left[ \frac{(1 - \delta) \hat{\theta} H(\hat{\theta}) q - 1}{qH(\hat{\theta})^2} \right]
\]

Notice from (95) that \( (1 - \delta) \hat{\theta} \geq 1 \). Also, integrating \( \phi_{i,1} \) over all \( \sigma_i \), we obtain

\[
\int \phi_{i,1} dH(\sigma_i) = \int_{\hat{\theta}} [(1 - \delta) \sigma_i - 1] dH(\sigma_i) \geq 0
\]

This implies that \( \int_{\hat{\theta}} (1 - \delta) \sigma_i dH(\sigma_i) \geq 1 - H(\hat{\theta}) \) and using (100) we get

\[
\frac{1}{q} = 1 - \int_{\hat{\theta}} (1 - \delta) \sigma_i dH(\sigma_i) \leq H(\hat{\theta})
\]

So that

\[
H(\hat{\theta}) q \geq 1
\]

Therefore, the first term in (105) is negative as \( (1 - \delta) \hat{\theta} \geq 1 \), the second term is also negative as \( H(\hat{\theta}) q \geq 1 \), and the third and final term is also negative as \( 1/q \leq H(\hat{\theta}) < 1 \) while the term is square bracket is positive. Therefore, the planner would like to set \( \hat{\theta} \) to the lowest possible value. This implies that \( (1 - \delta) \hat{\theta} = 1 \) (as long as \( \int_{1/(1-\delta)} (1 - \delta) \sigma_i dH(\sigma_i) < 1 \)).

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