Operational–risk Dependencies and the Determination of Risk Capital

Stefan Mittnik
Chair of Financial Econometrics, LMU Munich & CEQURA
finmetrics@stat.uni-muenchen.de

Sandra Paterlini
EBS Universität für Wirtschaft und Recht & CEQURA
sandra.paterlini@ebs.edu

Tina Yener
Linde AG & CEQURA
tina.yener@linde.com

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The heterogeneity of Operational Risk leads to the regulatory requirement of a separate modeling within 56 event–type/business-line combinations.

<table>
<thead>
<tr>
<th>Event Types</th>
<th>Business Lines</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>Corporate Finance</td>
</tr>
<tr>
<td>Internal Fraud</td>
<td>$L_{1,1}$</td>
</tr>
<tr>
<td>.</td>
<td>.</td>
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<tr>
<td>Execution, Delivery &amp; Process Management</td>
<td>$L_{7,1}$</td>
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</tbody>
</table>
Operational Risk: Total Risk Capital

The quantity of interest is

\[ \text{VaR}_{.999}(L) = \text{VaR}_{.999} \left( \sum_{i=1}^{56} L_i \right) ; \]  

(1)

clearly, it is influenced by dependencies among cells \( i \) and \( j \).

However, Basel II prescribes to calculate **Total Risk Capital** as

\[ \text{TRC} = \sum_{i=1}^{56} \text{VaR}_{.999}(L_i) ; \]  

(2)

Only under certain qualifying conditions, banks may explicitly model dependencies.
VaR and Subadditivity

It can be shown (Frachot et al., 2004) that for the case of comonotonic risks,

$$\text{VaR}^\text{co}_\alpha(L_i + L_j) = \text{VaR}_\alpha(L_i) + \text{VaR}_\alpha(L_j).$$  \hspace{1cm} (3)

Comonotonicity translates into perfect positive correlation in the elliptical (and thus, also the popular Gaussian) world.

For elliptical distributions, the sum of the single VaRs provides an upper bound and thus a worst–case scenario for $\text{VaR}_\alpha(L)$,

$$\text{VaR}_\alpha(L_i + L_j) \leq \text{VaR}_\alpha(L_i) + \text{VaR}_\alpha(L_j).$$  \hspace{1cm} (4)
VaR and Subadditivity

However, for non-elliptical distributions, it may happen that

\[ \text{VaR}_\alpha(L_1 + L_2) > \text{VaR}_\alpha(L_1) + \text{VaR}_\alpha(L_2), \]  \hspace{1cm} (5)

the reason being the lack of subadditivity of the VaR measure (Artzner et al., 1999).

Does this mean that banks may not be rewarded for a more realistic dependency modeling by a decrease in risk capital, but instead be punished by an increase?

Yes! (see, e.g., Embrechts et al., 2002)

But, is this practically relevant?
Aim of our Analyses

Based on $n = 60$ observations of monthly aggregate losses from the Italian DIPO $^1$ database, we aim at evaluating

$$\text{VaR} (L_i + L_j) - (\text{VaR} (L_i) + \text{VaR} (L_j)) = \text{TRC}$$

for different cells $i$ and $j$ of the event–type/business–line matrix.

This task is non–trivial, because it means analyzing the 99.9% quantile of a distribution estimated from a small sample with extreme data under consideration of dependencies.

To model dependencies, we focus on Correlation, Copulas and Nonparametric Tail Dependence measures.

$^1$http://www.dipo-operationalrisk.it
The well-known fact that linear correlation is prone to extremes is quickly revealed by the data.
For example, for event type combination (2. External Fraud; 5. Damage to Physical Assets), as the two most extreme observations drop out of the sample, correlation becomes negative.

\[ \rho_{2,5} = 0.0258 \]

\[ \rho_{2,5} = -0.1144 \]
Linear (Pearson) Correlation

Similarly, for event type combination (3. Employment Practices & Workplace Safety; 4. Clients, Products & Business Practice):

01/03–12/07 (entire sample): 01/03–04/06 (2/3 of sample):

\[ \rho_{3,4} = 0.5284 \]

\[ \rho_{3,4} = 0.5882 \]

If we further remove the most extreme observation between 01/03 and 04/06, the correlation decreases to \( \rho_{3,4} = 0.3232 \).
The well-known sensitivity of linear correlation with respect to extremes leads to substantial variations, depending on the sample size.

Its inability to capture possible nonlinear dependency structures provides another important reason for discarding linear correlation as a reliable measure of dependency.

Rank correlations were also considered but not found to lead to considerably more stable results.
Instead of boiling down dependency into one single number, copulas contain the dependence *structure* of a joint distribution.

The central theorem of copula theory can be traced back to Sklar (1959) and summed up by

\[
F_{i,j}(\ell_i, \ell_j) = C(F_i(\ell_i), F_j(\ell_j)),
\]

where \( C \) denotes the copula of \( L_i \) and \( L_j \) and \( U_i, U_j \sim \text{Unif}(0, 1) \).
Tail dependence accounts for—possibly nonlinear—dependence among extremes and thus overcomes one drawback of correlation.

Different copulas imply different tail dependence structures.

Gaussian copula: no tail dependence

Gumbel copula: upper tail dependence

Clayton copula: lower tail dependence
We fit alternative parametric copulas (i.e.: Gaussian, Student–t, Gumbel and Clayton).

<table>
<thead>
<tr>
<th></th>
<th>ET 2</th>
<th>ET 3</th>
<th>ET 4</th>
<th>ET 5</th>
<th>ET 6</th>
<th>ET 7</th>
</tr>
</thead>
<tbody>
<tr>
<td>ET 1</td>
<td>0.136</td>
<td>0.275</td>
<td>0.243</td>
<td>0.147</td>
<td>0.182</td>
<td>0.251</td>
</tr>
<tr>
<td>ET 2</td>
<td>0.176</td>
<td>0.235</td>
<td>0.020</td>
<td>0.000</td>
<td>0.115</td>
<td></td>
</tr>
<tr>
<td>ET 3</td>
<td></td>
<td>0.574</td>
<td>0.318</td>
<td>0.272</td>
<td>0.297</td>
<td></td>
</tr>
<tr>
<td>ET 4</td>
<td></td>
<td></td>
<td>0.359</td>
<td>0.353</td>
<td>0.154</td>
<td></td>
</tr>
<tr>
<td>ET 5</td>
<td></td>
<td></td>
<td></td>
<td>0.037</td>
<td>0.000</td>
<td></td>
</tr>
<tr>
<td>ET 6</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>0.218</td>
</tr>
</tbody>
</table>

Upper tail dependence coefficient implied by the *Gumbel* copula parameter estimates.
Copulas: Results

- Copula fitting suggests that some event type combinations are characterized by tail dependence, while others are not; i.e. (3;4) exhibits tail dependence, while (2;5) do not.
- However, we do neither find an overall “best–fitting” copula, nor can we exclude any copula family considered.
- Again, the availability of a small data set affects the stability of estimation results.
- We also consider Nonparametric Tail Dependence measures and empirical results support the presence of quantile dependence for (3;4) and its absence for (2;5).
Now, we want to assess the effects of realistic dependency structures on risk–capital estimates.

To this end, we estimate 250 99.9% VaR figures per model and event type combination, using different numbers of replications.

For each event type combination, we use the copula parameter values obtained from Maximum Likelihood estimation.

For the margins, a lognormal distribution was fitted and is used here to derive risk–capital estimates.
We consider

\[
\text{VaR}.999(L_i + L_j) - (\text{VaR}.999(L_i) + \text{VaR}.999(L_j)) \\
(\text{VaR}.999(L_i) + \text{VaR}.999(L_j))
\]  

(8)

under two different assumptions:

1. the Gaussian copula for all event type combinations,
2. the “worst-case” copula, i.e., that copula yielding the highest tail/quantile dependence coefficient for the respective event type combination.
Range of Risk–Capital Estimates: Boxplots

Range of simulated risk capital changes with Gaussian copula.

$B_{rc} = 10,000$

$B_{rc} = 50,000$

$B_{rc} = 100,000$
Range of Risk–Capital Estimates: Boxplots

Range of simulated risk capital changes with “worst–case copula.”

\[ B_{rc} = 10,000 \quad B_{rc} = 50,000 \quad B_{rc} = 100,000 \]
Bounds on Risk–Capital Estimates

Obviously, increasing the number of replications for VaR calculations narrows the range of possible risk–capital estimates.

It is, thus, not clear which part of a change is due to the subadditivity problem, and which one is due to computational issues.

A natural question is then: What could be the worst capital estimate? Statistically, this means to study whether there are theoretical bounds on VaR.

This topic has been treated, for example, by Makarov (1981) and Frank et al. (1987), and recently by Embrechts and Puccetti (2006).
The Fréchet–Höffding bounds (Fréchet, 1951; Höffding, 1940) apply to any $n$–dimensional copula, i.e.,

$$\max(u_1 + \ldots + u_n - n + 1, 0) \leq C(u) \leq \min(u) .$$

(9)
The tightness of the bounds on VaR depends on the dependence assumption.

We evaluate upper and lower bounds for three scenarios.

1. $C_0 = C_1 = C_\ell$: We do not use any restriction on the dependence structure and thus use the lower Fréchet bound, $C_\ell$.

2. $C_0 = C_1 = u_i u_j$: We assume that $C \geq u_i u_j$, that is, we have positive quadrant dependence (PQD).

3. $C_0 = C_{CS}^{\hat{\theta}_{i,j}}$, $C_1 = C_C^{\hat{\theta}_{i,j}}$: We take the Clayton Survival copula as lower bound, using the parameter values estimated for the DIPO data. For the survival copula of $C_1$, we accordingly assume the Clayton copula with respective parameter values.
Bounds on Risk–Capital Estimates: Boxplots

\[ B_{rc} = 10,000 \quad B_{rc} = 50,000 \quad B_{rc} = 100,000 \]

Relative variations in simulated risk capital and theoretical bounds based on a Gaussian copula.
Bounds on Risk–Capital Estimates: Boxplots

Relative variations in simulated risk capital and theoretical bounds based on a “worst–case copula.”

\[ B_{rc} = 10,000 \quad B_{rc} = 50,000 \quad B_{rc} = 100,000 \]
Bounds on Risk–Capital Estimates: Results

- Risk–capital estimates may increase when departing from the comonotonicity assumption. However,
  - this effect depends on the presence of extremal (tail/quantile) dependence;
  - such an increase may as well be caused by an insufficient number of replications in the simulation of losses.
- Theoretical bounds may help to assess which part of the change in risk capital is due to computational effects. The more restrictive the dependence assumptions used in deriving these bounds the more helpful they are.
Conclusion

- The question whether risk–capital estimates may increase/decrease compared to the comonotonicity case crucially depends on the presence of tail dependence and the ellipticity of the multivariate distribution.

- Simple methods, as correlations, may not lead to a complete and/or reliable picture of dependencies in operational risk losses. More sophisticated methods, such as copulas, could provide relevant information about dependencies and their effect on risk capital estimates.

- Risk–capital estimates may increase when departing from the comonotonicity assumption. Theoretical bounds on VaR may help to assess which part of the change in risk capital stems from effects due to the computational setup.

- Serious efforts towards improving database for operational–risk losses should be undertaken.
Further Research

- What is the effect of modelling multivariate dependence (beyond the bivariate case) on Total Risk Capital?
- Robust and stable estimation of risk capital by maximum entropy methods
Contacts

Prof. Stefan Mittnik, PhD  
Chair of Financial Econometrics, LMU Munich  
Akademiestr. 1/I, 80799 Munich, Germany.  
Tel. +49 (0) 8921803224, Fax. +49 (0) 8921805044  
Email: finmetrics@stat.uni-muenchen.de

Prof. Sandra Paterlini, PhD  
Chair of Financial Econometrics and Asset Management  
EBS Universität für Wirtschaft und Recht, Gustav-Stresemann-Ring 3 65189 Wiesbaden, Germany.  
Tel.: +49 (0) 611 7102 1217 ; fax: +49 (0) 611 7102 1217  
Email: sandra.paterlini@ebs.edu

Dr. Tina Yener  
Linde AG, Klosterhofstrasse 1, 80331 Munich, Germany  
Tel.: +49 (0) 89 35757 1614, Fax: +49 (0) 89 35757 1605,  
Email: tina.yener@linde.com
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