Dealing with Maturity: Solving for Optimal Fiscal Policy in the Case of Long Bonds

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Abstract

We study how to model optimal fiscal policy in the case where the government issues long bonds. We propose a flexible numerical method that can be used to solve this problem at low computational cost by substantially reducing the required state space. This numerical method enables us to maintain a rich economic structure and avoids restrictive assumptions or approximations about the economy, bond markets, maturity structure or the nature of bonds that other authors have had to make to consider this problem. Our computational method has much wider applicability than just Ramsey models of debt management. We analyse the reasons behind the complexity of optimal fiscal policy with long bonds and show that it arises from the temptation to manipulate interest rates and taxes to reduce funding costs in the case of non-zero initial debt. In other words, debt management concerns override tax smoothing considerations. We therefore propose an alternative means of simplifying the problem by considering a model of independent powers where interest rates are set by a monetary authority and debt issued by the fiscal authority. Finally the introduction of long bonds raises important issues about what the government does at the end of every period with outstanding debt. We show how to model the case of no buyback, where debt is left to maturity. We once again find governments with an incentive to manipulate interest rates and taxes and for debt management to override tax smoothing.

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1 Introduction

As the current European sovereign debt crisis emphasises the maturity structure of government debt is a key variable. Table 1 shows the average maturity of outstanding government debt for a variety of countries and displays clear differences across nations. Any theory of debt management needs to explain the costs and benefits for optimal fiscal policy in varying the average maturity structure in this way. However varying the maturity of government debt raises a number of computational and
modelling issues. In particular modelling optimal fiscal policy in the case of long bonds requires a large and computationally demanding state space and raises the issue of what the government does at the end of every period with the outstanding stock of debt. In this paper we show how to navigate these problems and examine the numerical impact of varying debt maturity on optimal fiscal policy and debt management.

The computational problem arises because under incomplete markets where the government issues a $N$ period bond and the economy is subject to $M$ shocks the required state space is $2N + M$. In the canonical case where governments issue one period bonds only (e.g Aiyagari et al (2002)) this is not an issue as with only government expenditure shocks considered the states space consists of three variables. However the dimension of the problem grows rapidly with maturity and given that both France and the UK issue fifty year bonds this makes for a very computationally demanding problem.

To reduce the computational complexity of solving for models with long bonds we propose two different solutions. The first is a computational method, based on the Parameterised Expectations Algorithm of den Haan and Marcet (1990), that enables us to solve for optimal fiscal policy in the case of a twenty year bond with a reduced state space of only four variables. This computational method has much wider relevance than solving for portfolio models with long bonds and can be used in any case where the model involves a large state space. The second solution to the problem of computational complexity involves identifying why solving the portfolio model with long bonds is so complex. As noted by Lucas and Stokey (1983), under the assumption of a Ramsey planner a government which inherits a non-zero level of initial period debt has an incentive to vary taxes and so twist interest rates in order to reduce future refinancing costs when debt matures. It is this incentive that leads to the large state space. We show how a model of independent powers, whereby an independent central bank sets interest rates and a debt management office focuses on optimal debt structure, is another way of solving for debt management with a much reduced state space.

A further problem when dealing with long bonds is what decision to make about outstanding debt at the end of each period. In the canonical case of one period bonds every period the government buys back all existing debt and then reissues. The same assumption has been made in models (e.g Angeletos (2002), Buera and Nicolini (2004)) with bonds of maturity greater than 1. In other words, governments issue $N$ period bonds and at the end of every period buy back the entire stock and then next period reissue debt as $N$ period bonds once again. However as shown in Marchesi (2004) governments rarely buy back outstanding debt before redemption. To quote the UK Debt Management Office (2003) "the UK’s debt management approach is that debt once issued will not be redeemed before maturity." In this case of no buyback when a government issues $N$ period bonds then at any moment in time the government has outstanding debt with maturity until redemption of $N, N - 1$ through to 1 year. The maturity profile of government debt is therefore much more complex with long bonds and no buy back and this will potentially impact debt management and fiscal policy. In this paper we propose methods to deal with both sets of these problems and so examine computationally the effect of varying the average maturity of government debt on optimal fiscal policy.

INSERT TABLE 1 HERE
Alternative approaches have been used to arrive at quantitative implications for incomplete markets and succeed either by making simplifying assumptions about the structure of the economy e.g Barro (2003) and Nosbusch (2008) ; offering a rich specification of the economy but limiting the length of bonds to be considered (Lustig, Sleet and Yeltekin (2009)\(^1\)), abstracting from issues of market incompleteness (Angeletos (2002)) or approximating long bonds by perpetuities with decaying coupon payments where the rates of decay mimic differences in maturity (Woodford (2001), Broner, Lorenzoni and Schmulker (2007), Arellano and Ramanarayanan (2008)). By contrast in this paper we outline a flexible method that can be used to solve numerically for a rich general equilibrium model with incomplete market features and arbitrary bond maturity.

The structure of the paper is as follows. Section 2 outlines our computational method - the condensed PEA whilst Section 3 shows the impact of the average maturity of government debt on optimal fiscal policy. Section 4 considers the case where the initial stock of government debt is non-zero and shows how the government has an incentive to manipulate interest rates. Section 5 outlines our model of independent powers whilst Section 6 considers the case of no buyback and a final section concludes.

2 Solving for Long Bonds

In this section we outline the structure of the incomplete market economy we study and the solution method we utilise to overcome problems with a large state space that inevitably occur when considering long maturity bonds. In essence we take the model of Aiyagari et al (2002) with one bond of maturity one period and extend to the case of longer maturities.

The economy produces a single non-storable good with a technology for every period \( t \) given by:

\[
c_t + g_t \leq 1 - x_t, \tag{1}
\]

where \( x_t, c_t \) and \( g_t \) represent leisure, private consumption and government expenditure respectively. We assume \( g_t \) to be stochastic and exogenous and is the only source of uncertainty in the economy.

The representative consumer has utility function:

\[
E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) \} \tag{2}
\]

and is endowed with one unit of time that it allocates between leisure and labour and faces a proportional tax rate \( \tau_t \) on labor income. The representative firm maximizes profits and both consumers and firms act competitively by taking prices and taxes as given. Consumers, firms and government observe all shocks up to the current period.

Agents can only borrow and lend in the form of a risk free \( N \)-period bond so that the government budget constraint is:

\[
g_t + p_{N-1,t} b_{N,t-1} = \tau_t (1 - x_t) + p_{N,t} b_{N,t} \tag{3}
\]

\(^1\) Lustig, Sleet and Yeltekin (2010) offer a very detailed model but restrict themselves to the case of \( n=7 \). The UK government classifies all debt of 7 years or less maturity as "short term debt".
where \( b_{N,t} \) denotes the number of bonds the government issues at time \( t \) and each bond pays one unit of consumption good in \( N \) periods time with complete certainty. The price of an \( N \) period bond at time \( t \) is \( p_{N,t} \) which has steady state value \( \beta^N \). For now we assume that at the end of each period the government buys back the existing stock of debt and then reissues new debt of maturity \( N \). In addition government debt has to remain within the upper and lower limits \( \underline{M} \) and \( \overline{M} \) so ruling out Ponzi schemes e.g

\[
\underline{M} \leq \beta^N b_{N,t} \leq \overline{M} \tag{4}
\]

The household’s problem is to choose stochastic processes \( \{c_t, x_t, b_{N,t}\}_{t=0}^{\infty} \) to maximize (2) subject to the sequence of budget constraints:

\[
c_t + p_{N,t} b_{N,t} = (1 - \tau_t) (1 - x_t) + p_{N-1,t} b_{N-1,t-1}
\]

with prices and taxes \( \{p_{N,t}, p_{N-1,t}, \tau_t\} \) taken as given. The household also faces debt limits analogous to (4), which we assume are less stringent than those faced by the government, so that in equilibrium, the household’s problem always has an interior solution.

The consumer’s first order condition in terms of consumption and leisure is:

\[
\frac{v_{x,t}}{u_{c,t}} = 1 - \tau_t \tag{5}
\]

and in terms of the risk free bond

\[
p_{N,t} = \frac{\beta^N E_t (u_{c,t+N})}{u_{c,t}}. \tag{6}
\]

### 2.1 The Ramsey problem

We assume the government has full commitment to implement the best sequence of (possibly time inconsistent) taxes and government debt and so solves the following Ramsey problem:

\[
\max_{c_t, x_t, b_{N,t}, p_{N,t}} E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) \}
\]

st

\[
g_t u_{c,t} + \beta^{N-1} E_t (u_{c,t+N-1}) b_{N,t-1} = (u_{c,t} - v_{x,t}) (c_t + g_t) + \beta^N E_t (u_{c,t+N}) b_{N,t}
\]

\[
\underline{M} \leq \beta^N b_{N,t} \leq \overline{M}
\]

To simplify the algebra we define \( S_t = (u_{c,t} - v_{x,t}) (A - x_t) - u_{c,t} g_t \) as the “discounted” surplus of the government and set up the Lagrangian as

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t [S_t + \beta^N u_{c,t+N} b_{N,t} - \beta^{N-1} u_{c,t+N-1} b_{N,t-1}] \\
+ \nu_{1,t} (\overline{M} - \beta^N b_{N,t}) + \nu_{2,t} (\beta^N b_{N,t} - \underline{M}) \}
\]

4
where $\lambda_t$ is the Lagrange multiplier associated with the government budget constraint and $\nu_{1,t}$ and $\nu_{2,t}$ are the multipliers associated with the debt limits.

Using the recursive contract approach of Marcet and Marimon (????) this problem can be rewritten as:

$$L = E_0 \sum_{t=0}^{\infty} \beta^t \left( u(c_t) + v(x_t) + \lambda_t s_t + u_{c,t} (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} + \nu_{1,t} (\bar{M} - \beta^N b_{N,t}) + \nu_{2,t} (\beta^N b_{N,t} - \bar{M}) \right)$$

for $\lambda_{-1} = \ldots = \lambda_{-N} = 0$. Note, that for $t = 0, \ldots, N-1$ the terms $u_{c,t} b_{N,t-N} (\lambda_{t-N} - \lambda_{t-N-1})$ are zero.

For $t \geq 1$ the first-order condition with respect to $c_t$ can be expressed as

$$u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) + u_{cc,t} (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,t-N} = 0$$

Taking the derivative of the Lagrangian with respect to $b_{N,t}$ gives

$$E_t (u_{c,t+N} \lambda_{t+1}) = \lambda_t E_t (u_{c,t+N}) + \nu_{2,t} - \nu_{1,t}$$

### 2.2 The Condensed PEA

Given this Ramsey problem the government faces occasionally binding constraints which prevents us from loglinearising the model and obliges us to use a global approximation solution method. We choose the Parameterized Expectation Algorithm of den Haan and Marcet (1990). However this Ramsey problem has a state space of $[\lambda_{t-1}, \ldots, \lambda_{t-N}, b_{N,t-1}, \ldots, b_{N,t-N}, g_t]$ with dimension $2N + 1$. In the case of Aiyagari et al (2002) where $N = 1$ the state space is small and PEA can be easily used. However in order to evaluate the case of government bonds of maturity 10, 20 or 50 years we would need to solve a model of dimension 21, 41 and 101 respectively which rapidly becomes intractable. To overcome this we outline below a modification of the PEA for linear approximations that enables models with a large state space to be solved with arbitrary degrees of accuracy whilst avoiding the curse of dimensionality.

Assume we wish to approximate the conditional expectation:

$$E_t \{ u_{c,t+N} \} .$$

and let the full state space for this problem be denoted $X_t = \{1, x_{t,1}, \ldots, x_{t,1+n+m}\}$ with dimension $n + m + 1 = 2N + 1$. Partition this vector into two component parts : a subset $X_t^{core} = \{1, x_{t,1}, \ldots, x_{t,n}\}$ which will be used to form a first initial approximation of the expectations and an omitted subset $X_t^{out} = \{x_{t,n+1}, \ldots, x_{t,n+m}\}$.
The condensed PEA then starts with the following linear approximation to the conditional expectation:

\[ E_t \{ u_{c,t+N} \} = X_t^{\text{core}} \beta. \]  \hspace{1cm} (10)

and uses this to simulate the model and converge on values for \( \beta \). Denote these converged values \( \beta_1 \).

This solution is of course based on a restricted set of state variables and so it is therefore necessary to see whether the omission of some state variables, in this case \( m \), affects the approximate solution. To test for this we propose the following procedure. First run a regression for every \( X_{out}^{i,t} \) in \( X_{out}^{t} \) (for \( i = 1 \ldots m \)) i.e

\[ X_{i,t}^{out} = X_t^{core} b_i^1 + X_{i,t}^{res,1}. \]

and calculate the residuals

\[ X_{i,t}^{res,1} = X_{i,t}^{out} - X_t^{core} b_i^1. \] \hspace{1cm} (11)

Next find \( \alpha_1 \) such that

\[ \alpha_1 = \arg \min \sum_{t=1}^T \left( u_{c,t+1} - X_t^{core} \beta_1 + X_t^{res,1} \alpha_1 \right)^2 \] \hspace{1cm} (12)

If the approximation based on the restricted steady state is accurate then the linear combination of omitted variables \( X_t^{res,1} \) should be insignificant and not add to the explanatory power of the regression. In other words if the adjusted \( R^2 \) of the two regressions is similar then the PEA based on the restricted steady state is sufficient.

If however the addition of the linear combination \( X_t^{res,1} \alpha_1 \) is significant then we need to start a new iteration approximating the conditional expectaton with:

\[ \left( X_t^{core}, \alpha_1 X_t^{res,1} \right) \beta_2 \]

where \( \beta_2 = \left( \beta_1^2 \right) \) with dimensions \( (n+2) \times 1 \)

\[ \beta_2 = \left( \begin{array}{c} \beta_1^2 \\ 1 \end{array} \right) \]

Once again we simulate the model using this approximation and converge on \( \beta_2 \). To test whether this augmented state space is sufficient we run the following regression for every \( X_{out}^{i,t} \) in \( X_{out}^{t} \) (for \( i = 1 \ldots m \)):

\[ X_{i,t}^{out} = \left( X_t^{core}, \alpha_1 X_t^{res,1} \right) b_i^2 \]

define the residuals

\[ X_{i,t}^{res,2} = X_{i,t}^{out} - \left( X_t^{core}, \alpha_1 X_t^{res,1} \right) b_i^2. \]
and then find $\alpha^2_{(m \times 1)}$ such that

$$\alpha^2 = \arg \min \sum_{t=1}^{T} \left( u_{c,t+1} - \left( X^\text{core}_t \alpha^1 X^\text{res,1}_t + X^\text{res,2}_t \beta^2 \right) \right)^2$$

Once again we see whether the addition of the linear combination of residuals is significant or whether the condensed state space is sufficient for the approximation to be accurate. The process continues until convergence is achieved with the desired level of tolerance or until all $m$ omitted variables have been reintroduced.

3 Optimal Fiscal Policy with Long Bonds

In this section we use the condensed PEA to solve for the optimal fiscal policy and examine how this changes as we extend the maturity of government debt from 1 period to 2, 5, 10, 15 and 20.

3.1 Benefits of the Condensed PEA

We calibrate our model to US data and assume the utility function:

$$c_t^{1-\gamma_1} + \eta x_t^{1-\gamma_2}$$

and choose $\beta = 0.98, \gamma_1 = 1$ and $\gamma_2 = 2$. We set $\eta$ such that the government’s deficit equals zero in the non-stochastic steady state and use the steady state condition to fix the fraction of leisure at 30% of the time endowment.

For the stochastic shock $g_t$ we assume a truncated AR(1) process e.g.:

$$g_t = \begin{cases} \bar{g} & \text{if } (1-\rho) g^* + \rho g_{t-1} + \varepsilon_t > \bar{g} \\ g & \text{if } (1-\rho) g^* + \rho g_{t-1} + \varepsilon_t < \bar{g} \\ (1-\rho) g^* + \rho g_{t-1} + \varepsilon_t & \text{otherwise} \end{cases}$$

We assume $\varepsilon_t \sim N(0,1.44)^2$, $g^* = 25$, with an upper bound $\bar{g}$ equal to 35% and a lower bound $g = 15\%$ of average GDP and $\rho = 0.95$. $\bar{M}$ is set equal to 90% of average GDP and $M = -\bar{M}$.

We need to approximate three expectations: $\Phi_\lambda = \beta^N E_t (u_{c,t+N} \lambda_{t+1})$, $\Phi_{ucN} = \beta^N E_t (u_{c,t+N})$ and $\Phi_{ucN-1} = \beta^{N-1} E_t (u_{c,t+N-1})$. We choose $X^\text{core}_t = \{1, \lambda_{t-1}, b^N_{t-1}, g_t\}$ and $X^\text{out}_t = \{b^N_{t-2}, ..., b^N_{t-N}, \lambda_t, ..., \lambda_{t-N}\}$.

To test for whether the condensed PEA has sufficient variables included for an accurate solution we use as our tolerance statistic:

$$\text{dist} = \frac{R^2_{\text{aug}} - R^2}{R^2}$$

where $R^2$ and $R^2_{\text{aug}}$ denote the goodness of fit of the original regression based on the condensed PEA and augmented with the linear combination of residuals respectively. We continue to add linear combinations until $\text{dist} \leq 0.0001$. Table 2 summarizes the number of linear combinations
needed for each maturity whilst Table 3 gives details and shows the number of linear combinations needed for each approximations and the $R^2$ and dist. The advantages of the condensed PEA are readily apparent: In nearly half the cases only three state variables are needed to solve the model and in no case are more than four variables required, with at most only one linear combination of omitted variables required to improve accuracy. Clearly the condensed PEA can be used to solve models with large state spaces with relatively small computational cost. Whilst we have focused on a case of optimal fiscal policy and debt management this methodology clearly has much broader applications to models with large state spaces.

![Table 2 and 3](here)

### 3.2 The Impact of Maturity

We initialize the model at 9 different initial conditions and simulate it for 5000 periods discarding the first 500 observations and we do this 1000 times per initial condition. Table 4 shows summary statistics for the economy and optimal fiscal policy in the case when the government issues one period bonds or ten period bonds and the government inherits initial period debt of zero. Figure 1 shows the impulse response functions for a selection of key variables. Reviewing these results reveals that varying the average maturity of debt makes very little difference to either the economy or fiscal policy. The summary statistics mostly differ only to the second or third decimal place and the impulse response functions are also very similar. The exception is the volatility of bond prices, debt issued and the market value of debt. Given that government expenditure shocks are persistent the price of long term bonds is more volatile than short term bonds and so, as a consequence, are the number of bonds the government has to issue and the market value of outstanding debt. **What I don’t understand here is why is the primary deficit so different for long bonds? Not only do I not get the intuition but I don’t get the arithmetic either as taxes, output and employment all seem the same?**

![Table 4 and Figure 1](here)

Another way of examining the impact of varying the average maturity of debt is to see whether this influences how close to the complete market outcome these incomplete market models can get. Marcet and Scott (2009) show that measures of relative persistence are a good way of assessing the extent of market incompleteness and so Figure 2 shows for various variables the measure:

$$P^k_y = \frac{Var(y_t - y_{t-k})}{kV\text{ar}(y_t - y_{t-1})}.$$

![Figure 2](here)

The closer to 0 this measure the less persistence the variable shows whereas the closer to 1 the measure the more the variable shows unit root persistence. Although the long bond model shows...
less persistence, suggesting that in the case of persistent government expenditure shocks issuing longer bonds helps provide more fiscal insurance, the difference between the two cases is minor. Given that taxes are distortionary we are not in a Modigliani-Miller world and how the government finances its expenditure can affect the real economy. However the fact that the differences across maturities are so small is perhaps not surprising. With the government only issuing one type of bond in each case and the yield curve showing broadly similar behaviour at different maturities the tax smoothing properties of debt issuance is achieved mainly through the role of debt as a buffer rather than through fiscal insurance. Further we are at this point following the rest of the literature in assuming that every period the government buys back all existing debt and then reissues. So although the government is issuing 10 period bonds it always buys them back after a year so is effectively always borrowing through one period debt, reducing the distinction between 1 period and 10 period bonds. We shall return to this issue in a later section.

4 Debt Management and Interest Rate Manipulation

The previous section showed that there were only minor differences in the behaviour of the economy and optimal fiscal policy as the government changed the maturity of debt it issued. However this result was based on a model where the government inherits in the initial period a zero debt level. By contrast Figure 3 shows the same impulse response functions when we assume $b_{N,-1} = \beta^N y^*$ where $y^*$ is the output in steady state.

In this case we can see a difference between the allocations with variables showing a blip in the first period for the case $N = 1$ and in period ten when $N = 10$. These twists in the impulse response functions reflect the term $u_{x,t} (\lambda_{t-N} - \lambda_{t-N+1}) b_{N,N+1}$ in (8). In the case of zero initial debt this term has no influence but in the presence of inherited debt the government has an incentive to manipulate interest rates to reduce the interest burden. When the government inherits initial period debt it benefits by lowering interest rates in the period when this debt matures and has to be rolled over. By setting high taxes at the maturity date the government achieves slower consumption growth and lower interest rates and so refinances on better terms. Note that this makes tax policy subordinate to debt management. Whereas for other periods debt management is structured to achieve tax smoothing in the case when the government inherits initial period debt it increases tax volatility to reduce the excess burden of taxation. Government forfeits tax smoothing in order to reduce average interest rate costs. In order to better understand the incentives of debt managers to manipulate interest rates we now consider a number of stripped down versions of our model so as to identify the different mechanisms at work. Doing so will lead us to a deeper insight as to why the state space is so large when government issues long term bonds.

4.1 A model without uncertainty: complete markets

Assume that government spending is constant, $g_t = \bar{g}$ and, as in Lucas and Stokey (1983), assume agents have access only to a one period risk free bond ($N = 1$). The budget constraint of the
government can be rewritten in period 0 as
\[ \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = b_{-1}^1 \]
\[ \sum_{t=0}^{\infty} \beta^t u_{c,t} \tilde{S}_t = b_{-1}^1 u_{c,0} \tag{13} \]
where \( \tilde{S}_t = (1 - \frac{u_{x,t}}{u_{c,t}}) (A - x_t) - g_t \) is the “non-discounted” surplus of the government. Equation (13) shows that if the government has some initial debt, \( b_{-1}^1 > 0 \), they use consumption in period 0 to decrease the burden of initial debt without increasing surpluses in all the following periods by implementing \( c_t < c_0 \) and \( \tau_t > \tau_0 \). Doing so lowers consumption in the initial period and reduces interest rates at \( t = 0 \).

Following the same reasoning with a long bond we have:
\[ \sum_{t=0}^{\infty} \beta^t \frac{u_{c,t}}{u_{c,0}} \tilde{S}_t = b_{-1}^N p_t^{N-1} \]
\[ \sum_{t=0}^{\infty} \beta^t u_{c,t} \tilde{S}_t = b_{-1}^N \beta^{N-1} \frac{u_{c,N-1}}{u_{c,0}} \]
\[ \sum_{t=0}^{\infty} \beta^t u_{c,t} \tilde{S}_t = b_{-1}^N \beta^{N-1} u_{c,N-1} \tag{14} \]

Again if the government has some initial debt, \( b_{-1}^1 > 0 \) they can reduce funding costs by manipulating consumption such that \( c_t < c_{N-1} \) by setting \( \tau_t > \tau_{N-1} \) and so reduce interest rates in \( N - 1 \) and lower funding costs for the \( N \) period bond.

### 4.2 A model with uncertainty in period 1: incomplete markets

The previous subsection focused on the impact of initial debt but abstracted from uncertainty. In this subsection we focus on the opposite - we set initial debt to zero and introduce uncertainty into our model, but only in the first period, by assuming the following structure for the expenditure shock:

\[
\begin{align*}
    g_t &= \bar{g} & &\text{for } t = 0 \\
    g_t &\sim U(g_{\min}; g_{\max}) & &\text{for } t = 1 \\
    g_t &= \bar{g} & &\text{for } t > 1
\end{align*}
\]

In other words government spending is constant in every period apart from the first where it is stochastic. Because uncertainty only exists in the first period the martingale condition implies future consumption is known in expectation so that :
\[
\begin{align*}
u_{c,t+N} \lambda_{t+1} &= \lambda_t u_{c,t+N} \\
\lambda_t &= \lambda_1 & &t > 1
\end{align*}
\]
The full first order conditions of the model are:

\[ u_{c,0} - v_{x,0} + \lambda_0 (u_{cc,0}c_0 + u_{c,0} + v_{xx,0} (c_0 + \bar{g}) - v_{x,0}) = 0 \quad \text{for } t = 0 \]

\[ u_{c,1} - v_{x,1} + \lambda_1 (u_{cc,1}c_1 + u_{c,1} + v_{xx,1} (c_1 + g_1) - v_{x,1}) = 0 \quad \text{for } t = 1 \]

\[ u^*_c - v^*_x + \lambda_1 (u^*_c c^* + u^*_c + v^*_x (c^* + \bar{g}) - v^*_x) = 0 \quad \text{for } t > 1 \text{ and } t \neq N \]

\[ u_{c,N} - v_{x,N} + \lambda_1 (u_{cc,N}c_N + u_{c,N} + v_{xx,N} (c_N + \bar{g}) - v_{x,N}) + u_{cc,N} [\lambda_0 - \lambda_1] b_{N,0} = 0 \quad \text{for } t = N \]

In this case we do not get constant consumption from period 1 onwards as this would violate (15) in period \( t = N \). Instead consumption jumps in periods 1 and 2, staying constant at a different level until period \( N + 1 \) where it changes again by reverting back to its previous value of \( N - 1 \). So for a given realisation of \( g_t \), we have:

\[ c_0 (\bar{g}) \quad \text{for } t = 0 \]

\[ c_1 (g_1) \quad \text{for } t = 1 \]

\[ c^* (\bar{g}) \neq c_1 (g_1) \quad \text{for } t > 1 \text{ and } t \neq N \]

\[ c_N (\bar{g}) \quad \text{for } t = N \]

If we assume that \( g \) can take only a finite number of realizations in period 1 then we can solve this model exactly. We assume the same utility function and value of the parameters as before. Moreover we assume \( N = 10 \) so that the asset available is a 10 period risk free bond. We let \( g \) take 51 possible values distributed uniformly around a mean \( g^* \). Tables 5 and 6 summarize the results for a model with a one period bond and a 10 period bond respectively. They show four different cases: when \( g_1 \) is higher or lower than \( g_0 \) and respectively when the government chooses to issue debt or buy private debt in period 0. The arrows show the behavior of variables with respect to their value the previous period.

When the government uses only a one period bond we only have an effect on the shock in the first 2 periods with the level of the shock determining the level consumption in period 1. In the model with a 10 period bond the shock affects not only consumption in period 1 and 2 but also consumption in period \( N \). When \( g_1 > g_0 \) the government knows they will have to run a deficit in period 1 because of the bad shock. If the government issues debt in period 0, they will want to decrease the burden of such debt by promising to decrease interest rates in period \( N \) and they will do the opposite if they buy private debt in period 0. When \( g_1 < g_0 \) the government knows that in period 1 they will be able to run a surplus and so they follow the opposite policy.

HERE TABLE 5 AND 6

5 Independent Powers

In an earlier section we showed how to solve incomplete market models of debt management with long maturities using a computational method that dramatically reduced the state space. In this section we show an alternative mechanism to solve these models in a simpler way. As was illustrated in the previous section the reason why the state space increases so dramatically with maturity in
incomplete market models is because of the desire of debt managers to twist interest rates. This
desire is captured in the Lagrange multiplier \( \lambda_{t-N} - \lambda_{t-N+1} \). To remove this temptation in this
section we assume the presence of a third agent, a monetary authority, that fixes in every period
the interest rates equal to:

\[
\begin{align*}
p_{N,t} &= \frac{\beta^N E_t (u_{c,t+N})}{u_{c,t}} \\
p_{N-1,t} &= \frac{\beta^{N-1} E_t (u_{c,t+N-1})}{u_{c,t}}.
\end{align*}
\]

In other words, the debt management authority in the Ramsey problem takes bond prices as given
and cannot directly manipulate them. In this case of independent powers the Lagrangian of the
Ramsey planner becomes

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \left\{ u(c_t) + v(x_t) + \lambda_t \left[ S_t + p_{N,t} b_{N,t} - p_{N,t-1} b_{N,t-1} \right] \\
+ \nu_{1,t} \left( M - \beta^N b_{N,t} \right) + \nu_{2,t} \left( \beta^N b_{N,t} - M \right) \right\}
\]

As a result the state space now only consists of the variables \( b_{N,t-1} \) and \( g_t \). The first order
condition with respect to consumption becomes

\[
u_{x,t} + \lambda_t \left( u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t} \right) + u_{cc,t} \lambda_t \left( p_{N,t} b_{N,t} - p_{N-1,t} b_{N,t-1} \right) = 0\]

and using the government’s budget constraint gives

\[
u_{c,t} - v_{x,t} + \lambda_t \left( u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t} \right) + u_{cc,t} \lambda_t \left( g_t - \left( 1 - \frac{v_{x,t}}{u_{c,t}} \right) (A - x_t) \right) = 0
\]

In the case of independent powers only the current level of \( \lambda_t \) matters, the government does not
have to keep track of all past promises in terms of consumption so the term \( (\lambda_{t-N} - \lambda_{t-N+1}) \) does
not appear anymore. What matters now is the level of deficit in every period not the level of debt
inherited from the past.

To see the impact of Independent Powers we calibrate the model as in Section 2 and consider the
case where \( N = 10 \) and \( b_{N-1} = 0.5y^*/\beta^N \). Figure 4 shows the impulse responses to a one standard
deviation shock to the level of government spending with the solid line showing the responses for
the model of independent powers and the dashed line of the standard model as in Section 2. As can be
seen the model of independent powers is not affected by the level of initial debt so no interest
rate twisting occurs. In this case debt management is subservient to tax smoothing.

HERE FIGURE 4 and TABLE 7

We can compare our independent powers model with one where debt managers engage in interest
rate twisting to better understand the magnitude of this effect. We simulate the independent powers
model and the standard model starting from 9 different initial conditions on debt 1000 times for
every initial condition. We simulated the model at different time horizons $T = 40$, $T = 200$ and $T = 5000$ discarding the first 500 periods. We calculated the standard deviation of taxes and we averaged it across simulations. We repeat the same exercise for $N = 2, 5, 10, 15, 20$. Figure 5 shows the results.

**HERE FIGURE 5**

In shorter sample periods the effect of twisting interest rates in connection with initial period debt is significant and provides a higher level of tax volatility. As we increase the sample size the initial period effect diminishes. This suggests that the model of independent powers may be advantageous for considering long term properties of the model but for a focus on short run samples or the impact of initial debt it may be misleading.

**6 No Buy Back**

We have shown in Section 2 how to solve computationally for optimal fiscal policy in the case of long maturity bonds. However in doing so we ignored an important issue that only arises as we increase $N$ from 1. With bonds of more than one period maturity the government has a choice to make at the end of every period. It can either buyback the $N$ period bonds it issued last period, which are now of course of maturity $N - 1$, and then reissue new $N$ period bonds. Alternatively it can leave the outstanding bonds to maturity (what we call no buyback) and issue new $N$ period bonds. In the case of buyback there are only $N$ period bonds outstanding. In the case of no buyback there exist bonds at all maturities between 1 and $N$ even though the government only issues $N$ period bonds. As shown in Marchesi (2004) it is normal practice for governments not to buyback debt - debt is issued and only bought back at maturity. Exactly why this is standard practice\(^2\) is beyond the scope of this paper but in this section we outline how to use our computational method to solve for optimal fiscal policy when debt managers do not buyback debt at the end of each period.

The economy is as before but in the case of no buyback the government budget constraint and debt limits are

\[
g_t + b_{N,t-N}^{NBB} = \tau_t (1 - x_t) + p_{N,t} b_{N,t}^{NBB}
\]

and

\[
M^{NBB} = \frac{M}{N} \leq b_{N,t}^{NBB} \leq \frac{M}{N} = \bar{M}^{NBB}
\]

(18)

(19)

The household’s problem is to choose stochastic processes $\{c_t, x_t, b_{N,t}^{NBB}\}_{t=0}^{\infty}$ to maximize (2) subject to the sequence of budget constraints:

\[
c_t + p_{N,t} b_{N,t}^{NBB} = (1 - \tau_t) (1 - x_t) + b_{N,t-N}^{NBB}
\]

with prices and taxes $\{p_{N,t}, \tau_t\}$ taken as given.

\(^2\) Conversations with debt managers suggest some combination of transaction costs, a desire to create liquid secondary markets at most maturities or worries over refinancing risk. This is the reason why we rule out a third possibility - governments choosing to only buy back a certain proportion of outstanding debt.
6.1 The Ramsey problem

The Ramsey planner solves the following problem:

\[
\max_{c_t, x_t, b_{N,t}^{\text{NBB}}, p_{N,t}} E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) \}
\]

st

\[
g_t u_{c,t} + b_{N,t-1}^{\text{NBB}} = (u_{c,t} - v_{x,t}) (c_t + g_t) + \beta^N E_t (u_{c,t+1} + b_{N,t}^{\text{NBB}}) - b_{N,t-1}^{\text{NBB}}
\]

\[
M_{\text{NBB}}^{\text{NBB}} \leq b_{N,t}^{\text{NBB}} \leq M_{\text{NBB}}^{\text{NBB}}
\]

We can set up the Lagrangian as

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t [S_t + \beta^N u_{c,t+1} b_{N,t}^{\text{NBB}} - b_{N,t-1}^{\text{NBB}}]
\]

\[
+ \nu_{1,t} (M_{\text{NBB}}^{\text{NBB}} - b_{N,t}^{\text{NBB}}) + \nu_{2,t} \left( b_{N,t}^{\text{NBB}} - M_{\text{NBB}}^{\text{NBB}} \right) \}
\]

where \( \lambda_t \) is the Lagrange multiplier associated with the government budget constraint, \( \nu_{1,t} \) and \( \nu_{2,t} \) are the ones associated with the debt limits.

Using a recursive approach once more this can be rewritten as

\[
L = E_0 \sum_{t=0}^{\infty} \beta^t \{ u(c_t) + v(x_t) + \lambda_t S_t + u_{c,t} (\lambda_{t-1} - \lambda_t) b_{N,t-1}^{\text{NBB}}
\]

\[
+ \nu_{1,t} (M_{\text{NBB}}^{\text{NBB}} - b_{N,t}^{\text{NBB}}) + \nu_{2,t} \left( b_{N,t}^{\text{NBB}} - M_{\text{NBB}}^{\text{NBB}} \right) \}
\]

for \( \lambda_1 = \ldots = \lambda_{N} = 0 \). The problem consists of \( 2N+1 \) states \( \left[ \lambda_{t-1}, \ldots, \lambda_{t-N}, b_{N,t-1}^{\text{NBB}}, \ldots, b_{N,t-N}^{\text{NBB}}, g_t \right] \).

For \( t \geq 1 \) the first-order condition with respect to \( c_t \) can be expressed as

\[
\begin{align*}
&u_{c,t} - v_{x,t} + \lambda_t (u_{cc,t} c_t + u_{c,t} + v_{xx,t} (c_t + g_t) - v_{x,t}) \\
&+ u_{cc,t} (\lambda_{t-1} - \lambda_t) b_{N,t-1}^{\text{NBB}} = 0
\end{align*}
\]

(21)

Taking the derivative of the Lagrangian with respect to \( b_{N,t}^{\text{NBB}} \)

\[
E_t (u_{c,t+1} + \lambda_t \lambda_{t+1}) = \lambda_t E_t (u_{c,t+1}) + \nu_{2,t} - \nu_{1,t}
\]

(22)

We have in this paper shown two ways of reducing the large state space that is an intrinsic feature of a model with long bonds – the condensed PEA and the model of independent powers. However in the case of no buyback the condensed PEA is preferable. With no buy back the state space with independent powers consists of \( \left[ b_{N,t-1}^{\text{NBB}}, \ldots, b_{N,t-N}^{\text{NBB}}, g_t \right] \) and so has dimension \( N+1 \) compared with 2 in the case with buyback. In the case of no buyback, independent powers does still lead to a significant reduction in the state space but not as large as with buyback. For bonds of maturity \( N > 10 \) and no buyback even the model of independent powers represents a considerable computational burden suggesting that the condensed PEA is more attractive in these circumstances.
6.2 The results

We calibrate the model as in the previous sections. In this case we need to approximate two expectations: \( \Phi_\lambda = \beta^N E_t (u_{c,t+N} \lambda_{t+N}) \), \( \Phi_{uc_N} = \beta^N E_t (u_{c,t+N}) \). We choose the core to be \( X_t = \{1, \lambda_{t-N}, b_{N,t-N}^{NBB}, g_t\} \). The state variables left out are \( X_t^{out} = \{b_{N,t-1}^{NBB}, \ldots, b_{N,t-N-1}^{NBB}, \lambda_{t-1}, \ldots, \lambda_{t-N-1}\} \), equal to \( 2(N-1) \) variables. We keep the same tolerance level as in the model with buy back. Table 8 summarizes the number of linear combinations we needed to approximate our expectations. Relative to the case where government buys back debt each period the required state space is larger - more linear combinations of residuals are needed - although the condensed PEA still dramatically reduces the state space.

Table 9 shows summary statistics for the model with no buyback and bonds of varying maturities. Broadly the results are similar although consumption and taxes show greater volatility now with long bonds and no buyback. Because debt is held to maturity each period the government now issues fewer bonds per period. Figure 6 shows the impulse response functions for the model in the case of zero initial debt, a 10 period bond and no buyback and compares them with the case of a one and 10 period bond and buyback. In the case of no buyback we can see that varying the maturity of the bond does affect optimal policy. This can be seen most clearly by examining the impulse response functions for tax rates.

Under buyback taxes show the smooth and highly persistent behaviour of a random walk as shown by Barro (1979). However with no buyback taxes behave differently. Instead taxes decline over the first \( N \) periods and then jump again and thereafter decline once more. We saw earlier that governments had the incentive to twist interest rates by setting \( \tau_n > \tau_{n-1} \) so that consumption growth between \( n-1 \) and \( n \) is low and so therefore are interest rates and refinancing costs. Because of the no buyback condition the debt manager now smooths this effect over each period to produce declining tax rates over the life of the bond. As can be seen this leads to a higher level of output (taxes are lower) and a larger deficit (lower taxes offset the higher output), the market value of debt shows greater persistence and the excess burden of taxation no longer follows a risk adjusted martingale as in Aiyagari et al (2002). The greater tax volatility and the non martingale behaviour are testimony to how in the case of no buyback debt management concerns shape the optimal fiscal policy.

7 Conclusions

This paper has had two interrelated aims. The first has been to take seriously the problems that occur when governments issue bonds of long maturity. The second has been to propose a general method for solving models with a large state space - the condensed PEA.
When governments issue bonds of maturity greater than 1 a number of additional considerations need to be considered. When governments issue long term bonds and inherit initial period debt they have an incentive to twist interest rates to minimise funding costs. It is this property that leads to the state space for an economy with a N period bond being of dimension 2N+1. Using the condensed PEA enables us to solve this model accurately with a much reduced state space. An alternative method we proposed to reduce computational complexity is to remove the ability of debt managers to twist interest rates by invoking a model of independent powers whereby a central bank determines interest and a debt manager issues debt. The model of independent powers is a specific solution to a problem where the government can affect prices. The condensed PEA is however a more general technique that is relevant to any model with a large state space.

The other significant issue in considering the case of long bonds is whether governments buyback each period the existing stock of debt. Clearly in practice governments do not buyback debt and we show how to model this and solve using the condensed PEA.

In the case of buyback our simulations suggest that in the case of zero initial debt there is little difference in fiscal policy or economic allocations as the maturity of debt is varied. Our main focus in this paper has been computational so we have not considered the robustness of this result to alternative specifications. Faraglia, Marcet, Oikonomou and Scott (2011) show in the context of a sticky price nominal model that the average maturity of debt exerts an important influence on fiscal policy.

In the case of buyback and nonzero initial debt the maturity of bonds does affect fiscal policy as governments have an incentive to twist interest rates in response to the inherited debt. As a consequence debt management concerns help shape the path of fiscal policy. This effect is even stronger in the case of no buyback where tax rates no longer follow the risk adjusted martingale process that pure tax smoothing considerations predict but show additional dynamics that impact on fiscal policy.

Whilst this paper has shown how to address problems caused by introducing long bonds into the analysis a number of further issues remain. We have throughout this paper assumed the government can issue only one bond but have varied its maturity. In order to fully understand debt management we need to consider the case when the government can issue several bonds of different maturity. Another important issue is to consider why governments do not buyback debt – presumably because of concerns over transaction costs. Extending the methodologies of this paper in this manner will enable us to provide a detailed numerical analysis of optimal debt management.
References


[16] Marcet, A and Marimon, R (????).


Economic Journal, 118, 477–498

Table 1 - Average Maturity Government Debt 2010

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<thead>
<tr>
<th>Country</th>
<th>Average Maturity (Years)</th>
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</tr>
<tr>
<td>Denmark</td>
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</tr>
<tr>
<td>Greece</td>
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</tr>
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</tr>
<tr>
<td>Switzerland</td>
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</tr>
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</table>

Source: OECD, The Economist

Table 2: Model with buyback - construction of a sufficient state space

| maturity | total # of states | size of the core* | # of linear comb. | \( \Phi_\Lambda \) | \( \Phi_{ucN} \) | \( \Phi_{ucN-1} \) |
|----------|-------------------|-------------------|-------------------|-------------------|-----------------|-----------------
| 1        | 3                 | 3                 | -                 | -                 | -               | -                |
| 2        | 5                 | 3                 | 0                 | 1                 | 0               | 0                |
| 5        | 11                | 3                 | 0                 | 1                 | 0               | 0                |
| 10       | 21                | 3                 | 0                 | 1                 | 0               | 0                |
| 15       | 31                | 3                 | 0                 | 1                 | 0               | 0                |
| 20       | 41                | 3                 | 0                 | 1                 | 1               | 0                |

*we do not count the constant

Table 3: Model with buyback - construction of a sufficient state space
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Table 5: 10 period bond model - uncertainty in $t = 1$

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<td>↓</td>
<td>↑</td>
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| $g$↓  | +        | ↑     | ↓   | =     | +                       | +                                   |
| $g$↓  | -        | ↑     | ↓   | =     | +                       | -                                   |

Table 6: 10 period bond model - uncertainty in $t = 1$

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<td>↓</td>
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| $g_1$↓ | +         | ↑     | ↓     | ↓     | +                       | +                                   |
| $g_1$↓ | -         | ↑     | ↓     | ↑     | +                       | -                                   |
Table 7: Independent Power Model with Different Maturities

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*we do not count the constant
### Table 9: No Buy Back Model with Different Maturities

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Figure 1: 1 and 10 Period Bond: Impulse Responses - $b_{N,-1} = 0$
Figure 2: 1 and 10 Period Bond: K-variances

Optimal model with buy back
K-variances: 1 period bond

Optimal model with buy back
K-variances: 10 period bond

Optimal model with buy back
K-variances: taxes
Figure 3: 1 and 10 Period Bond: Impulse Response - $b_{N,-1} = 0.5y^*/\beta^N$
Figure 4: Standard Model and the Model of Independent Powers: Impulse Response

\[- b_{N,-1} = 0.5y^* / \beta^N \]
Figure 5: Standard Model and the Model of Independent Powers: Tax Volatility
Figure 6: Standard Model with and without Buy Back: Impulse Response - $b_{N,-1} = 0$
Figure 7: Buy Back and no Buy Back Model: Tax Volatility

Optimal model with and without buy back
Variability of taxes

- t=40
- t=200
- long run