Imperfect Knowledge about Asset Prices and Credit Cycles*

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Abstract

We develop an equilibrium model with collateral constraint in which rational agents possess imperfect knowledge and learn about collateral price process. Bayesian updating of beliefs by agents could endogenously generate boom and bust in collateral prices and largely strengthen the role of collateral constraint as amplification mechanism through the interaction of agents’ belief, collateral prices and credit limit. Over-optimisms or pessimisms are fueled when surprise in price expectations is interpreted partially by the agents as permanent change in the parameters governing collateral price process and are validated by subsequently realized prices. We show the model could quantitatively account for recent US boom-bust in house prices, debt and aggregate consumption dynamics during 2001-2008. We also show a leveraged economy with higher steady state leverage ratio or elasticity of users’ cost with respect to borrowers’ collateral holding is more prone to self-reinforcing learning dynamics.

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“At some point, both lenders and borrowers became convinced that house prices would only go up. Borrowers chose, and were extended, mortgages that they could not be expected to service in the longer term. They were provided these loans on the expectation that accumulating home equity would soon allow refinancing into more sustainable mortgages. For a time, rising house prices became a self-fulfilling prophecy, but ultimately, further appreciation could not be sustained and house prices collapsed.”

Bernanke, Speech, Monetary Policy and the Housing Bubble, at the Annual Meeting of the American Economic Association, Atlanta, Georgia, January 3, 2010

1 Introduction

In the recent decade the US economy has witnessed massive run-up and then crash of house prices, large fluctuation of housing value, as well as the remarkable role of the interaction between housing markets and credit markets in aggregate fluctuation. Real house price increased considerably in the decade before recent financial crisis, as indicated by the index\textsuperscript{1} depicted in the upper panel of figure 1. It displayed relative smaller variability before the year 2000 and increased by about 40% from 2000 to 2006. House price booms were associated with widening household debt/GDP ratio. As can be seen from the lower panel of figure 1, the ratio of net credit market assets\textsuperscript{2} of US household and non-profit organizations to GDP changed moderately before the year 2000 but fell by roughly 90%, from −40 to −75 percent from 2000 to 2006. Figure 2 displays US aggregate consumption growth\textsuperscript{3} during recent decade. Consumption comoved strongly and positively with house prices during recent housing boom and bust. Consumption grew at about 3% annual rate with a cumulated growth 17.9% from 2000 to 2006, while its growth dropped sharply when house price started to reverse.

A significant body of research that are launched attempting to understand the recent house price dynamics, the role of housing sector in aggregate fluctuation and/or of various policies are built upon an influential work by Kiyotaki and Moore (1997, henceforth KM) with collateral constraint. Due to limited contract enforcement, economic agents have limited ability to borrow and loans are secured by collateral assets, e.g., land. Borrowers’ credit limit are affected by the prices of the collateralized assets, meanwhile these prices are affected by the size of credit limits. KM argue that

\textsuperscript{1}The data is taken from the OECD. Its definition is nationwide single family house price index. The real house price index is the nominal house price index deflated by CPI price index. It is normalized to a value of 100 in 2000. The price-to-rent ratio and price-to-income ratio display a similar pattern.

\textsuperscript{2}The data is from the Flow of Funds Accounts of the U.S. provided by the Board of Governors of the Federal Reserve System.

\textsuperscript{3}The data is from Federal Reserve System. It is the growth rate of Real Personal Consumption Expenditures.
Figure 1: US real house prices and Net Credit Market Assets

dynamic interaction between asset prices and credit limits may serve as a transmission mechanism by which shocks persist, amplify and spread.

Despite the critical role in recent macroeconomic turmoil, such massive run-up of house prices and large fluctuation of housing value are extremely difficult to be generated in most existing optimizing-agents DSGE models with housing sector and collateral constraints. These models with collateral constraints typically assume agents could rationally foresee the future collateral prices associated with any possible contingency. Therefore, collateral prices are tightly linked to the fundamentals of the economy, while the latter have relative smaller variability. The role of collateral constraint in aggregate fluctuation in these models may be downplayed when they do not generate sufficient variability in collateral prices as observed in the data.

From theoretical point of view, rationally anticipating collateral prices associated with any possible contingency requires very strong informational assumption on the agents. In a basic asset pricing model with heterogenous belief and standard incompleteness, Adam and Marcet (2011) shows that the Rational Expectation Equilibrium needs at least that all agents’ preferences and beliefs must be common knowledge and the pricing equation must be common knowledge.

Contrast with these literatures with collateral constraint and complete information, in this paper we present a model with collateral constraint allowing economic agents possess imperfect knowledge and learn optimally about the stochastic process driving collateral prices. The interaction of agents’ belief and collateral constraint largely amplifies and propagates fundamental shocks and could give rise to boom and bust in collateral prices. In addition, the role of collateral constraints as amplification
mechanism in aggregate fluctuation is largely strengthened due to more variability of collateral prices. We show the model could quantitatively account for the recent boom-bust in house prices, debt and aggregate consumption dynamics during 2001-2008 following a sequence of real interest rate reduction after year 2000. We also show a leveraged economy with higher steady state leverage ratio or elasticity of users’ cost with respect to borrowers’ collateral holding is more prone to self-reinforcing learning dynamics.

Agents in our model are assumed not knowing other agents’ belief about future collateral prices and discount factors. Relaxation of such informational assumptions leads to agents are uncertain about the mapping from fundamentals of the economy to collateral prices and cannot simply derive the equilibrium distribution of market prices. Instead, following Adam and Marcet (2011), we assume agents in our model are “Internal Rational”, maximizing infinite horizon utility under uncertainty but hold fully specified and dynamically consistent set of subjective belief about the process governing collateral prices. The equilibrium distribution of collateral prices will not necessarily be equal to agents’ subjective probability distribution.

Specifically, we assume that agents entertain subjective belief system that collateral prices depend on borrowers’ collateral holding, which has the same expression as the collateral price process in the Stochastic Rational Expectation Equilibrium of our model. Furthermore, we assume agents are uncertain about the parameters in their belief system. Optimal behavior implies that they apply Bayes’ law to market outcomes to update their beliefs.

Significantly different dynamics of prices and quantities in the learning model arises from following reasons. Firstly, transitory fluctuation in collateral prices is partially interpreted by the agents as permanent change of parameters governing collateral prices process, which fuels agents’ over-optimism or pessimism about future collateral prices.
and the liquidation value of collateral. Secondly, erroneous optimism or pessimism is partially validated and sustained by subsequently realized prices for many periods. The interaction of agents’ belief, credit limit and asset prices generate large additional propagation and could fuel boom and bust in prices, i.e., prices are temporarily delinked from fundamentals.

The dynamic interaction of agents’ belief, collateral prices and credit limit in our learning model could be understood by considering a positive shock to collateral prices, e.g., a productivity shock or real rate reduction. Positive surprise in collateral prices leads to updating of agents’ belief about the parameters governing the collateral price process. Agents’ expectation about future collateral prices becomes more optimistic than under RE. Based on the optimism about expected liquidation value of collateral, collateral constraint is relaxed and larger loans are granted by lenders. With larger borrowing capacity, borrowers will have larger demand on the collateral, which boosts collateral prices further up. The realized collateral prices partially validate agents’ belief and lead to further optimism. The capital gain of collateral holding outweighs temporarily the rise of users’ cost of collateral, which generates momentum and self-reinforcing dynamics in collateral prices, collateral holding and lending. Large shift of collateral to more productive borrowers facilitates prolonged periods of expansion of aggregate output and consumption.

Collateral price booms will come to an end due to endogenous model dynamics or adverse fundamental shocks. As borrowers acquire more and more collateral, the collateral holding of lenders decreases due to fixed supply. Since lenders themselves have decreasing return to scale technology to produce, the opportunity cost of lending to borrowers rises in equilibrium. When the capital gain falls short of users’ cost of collateral, borrowers start to reduce their holding of collateral. The rising collateral price cannot be sustained and eventually the optimism about future collateral price will be choked off. When collateral price falls short of expectation, agents becomes pessimistic and revise their belief downward. Credit limit is tightened based on pessimism about future liquidation value of collateral. The realized collateral prices reinforce the initial pessimism and display negative momentum. The realized prices and quantities decline faster toward and eventually converge to the steady state level.

Previous literatures, such as Timmerman (1996), Carceles-Poveda and Gannitsarou (2008), do not find large effect self-referential learning in explaining asset prices volatilities observed in data when agents learn about parameters linking price and dividend/capital in an endowment or production economy without collateral constraint. Even though agents learn such link, our model could generate large learning effect due to a key ingredient, i.e., collateral constraint and a key property of model dynamics, i.e., momentum in beliefs and hence in actual prices. Our model with collateral constraint has richer asset pricing equation. Current collateral prices depend not only on current belief but also on inherited debt holding, which in turn depends on expecta-

\[ \text{The users’ cost could be thought as opportunity cost of using capital. In equilibrium it is equal to lenders’ marginal productivity of land.} \]
tion about current price formed previously. Importantly, collateral price depends on change of beliefs. Increase in optimism (pessimism) in beliefs might lead to high (low) realization of collateral prices, which leads to further optimism (pessimism). Actual collateral prices may display momentum in the learning economy.

Whether the learning economy displays momentum in beliefs depends critically on key parameters in the model. We find that momentum in beliefs arises more easily in the learning economy with higher steady state leverage ratio or elasticity of users’ cost with respect to borrowers’ collateral holding. Given agents’ expectation formed previously, more optimistic expectation about future collateral prices will arise with larger leverage ratio or elasticity after a positive shock to prices. The more the upward revision of agents’ expectation is, the more easily that users’ cost stays below capital gain. Learning economy with higher steady state leverage ratio or elasticity w.r.t. borrowers’ collateral holding is more prone to generate self-reinforcing learning dynamics in prices and quantities.

The Rational Expectation Equilibrium for collateral price could be equivalently formulated as (log deviation of) collateral prices depends on productivity shock and previous collateral prices. Given specifying agents’ belief system is an open issue in this literature, we also consider an alternative belief such that agents perceive (log deviation of) collateral prices follow a mean reversion process, i.e., depend on past collateral prices and productivity shocks. Again agents are assumed to be uncertain and learn about the mean and persistence of (log deviations of) collateral prices. We find resulted learning dynamics is similar to the previous one due to interaction of agents’ belief, credit limit and actual outcomes. Momentum in belief is always present for all admissible parameters in the learning economy with this alternative belief system. In addition, momentum generated under such belief system is stronger than under previous belief system.

Adam, Marcet, and Nicolini (2009), Adam and Marcet (2010) show that relaxation of the strong informational assumption leads to that asset pricing models with learning about price/return behavior could quantitatively replicate major stock pricing facts, generating boom and bust in stock prices and matching agents’ return expectation. This paper extends these work with a collateral constraint limiting household borrowing.

Adam, Kuang, and Marcet (2011, henceforth AKM) account for heterogenous G7 house prices and current account dynamics over 2001-2008 following real rates reduction after 2000 and then increase at 2006 in a model with housing collateral constraint and learning about house price growth by households. This paper differs with AKM along several important dimensions. Firstly, distribution of collateral between borrowers and lenders with different productivities plays an important role in our model generating output and consumption amplification but lost in the AKM specification. Secondly, we work with different subjective belief system of agents. Once we allow agents’ beliefs could have small deviation from REE beliefs, it is very interesting to work out the model implication of other near-rational belief specifications. Finally, we also analyze the role of leverage ratio when allowing agents have near-REE beliefs.
The rest of the paper is structured as follows. The next section discusses related literature. Section 3 presents our benchmark model, agents’ optimality conditions and the RE equilibrium. In section 4, we discuss the equilibrium with imperfect knowledge, belief specification and optimal learning behavior of agents. The mechanism of our learning model is inspected in section 5. We examine an extension of the model and a modification of agents’ belief system in section 6. Quantitative results are presented in section 7. Section 8 examines an alternative subjective belief system that collateral prices follow mean reversion process. Section 9 concludes.

2 Related Literature

Prior to recent financial crisis, collateral constraints have been studied as amplification mechanism transforming relative small shocks into large output fluctuations. Examples are Kiyotaki and Moore (1997, henceforth KM), Kiyotaki (1998), Kocherlakota (2000), Krishnamurthy (2003), Cordoba and Ripoll (2004). Lustig and Van Nieuwerburgh (2005) study the role of house prices and housing collateral for the pricing of stocks. Policy analysis and welfare analysis are conducted in models with collateral constraint, such as Iacoviello (2005), and Cooley, Marimon, and Quadrini (2005).

More recently, Liu, Wang and Zha (2011) examine the role of different shocks in an estimated DSGE model in which entrepreneurs use land and capital as collateral for borrowing. They identify housing preference shocks as the most important driving force among all shocks in their model. They also provide a mechanism with competing demand of land between household and business sector, which largely propagates this shock and generates large amplification and positive comovement of land prices and business investment as observed in the data.

Ferrero (2011) considers the role of progressive relaxation of credit constraints, departure of interest rate from a standard monetary policy rule and of exchange rate peg in recent US housing boom in a model with collateral constraints limiting domestic households’ borrowing. He argues the model could account for a sizable portion of house prices booms and current account deficits. Contrast with these literature with complete information, we allow economic agents have imperfect knowledge about collateral price process.

Models with imperfect information are developed to understand dynamics of house prices and aggregate activities after recent housing booms and subsequent financial crisis, given large variability of prices and quantities observed in the data is hard to be reconciled with relatively smaller variability of fundamentals under full information rational expectation models.

Boz and Mendoza (2011) studies the role of learning about the riskness of a new financial environment in a model with collateral constraint. Realizations of states with high (low) leverage to borrow could lead to agents overly optimistic (pessimistic) about the probability of persistence of high-leverage (low-leverage) regime. The interaction of this learning friction and Fisherian deflation mechanism accounts for a large portion of
The interaction of revision in growth expectation and borrowing constraint amplifies shocks. Burnside, Eichenbaum and Rebelo (2011) presents a model in which a temporary house price boom emerges from infectious optimism that eventually dissipates once investors become increasingly certain about fundamentals.

Our paper is related to literatures which try to explore the role of shifting expectations as a source of business cycle fluctuation, in particular based on learning dynamics. Huang, Liu and Zha (2009) study the implications of adaptive expectations in a standard growth model. The differences with RE are mainly driven by dampened wealth effect and the strengthened intertemporal substitution effect. They find that adaptive expectations might be an important source of frictions that amplify and propagate technology shocks and seem promising for generating plausible labor market dynamics. Eusepi and Preston (2010) studies infinite horizon learning behavior about future return to capital and labor amplifies and propagates technology shocks in a real business cycle model. Erroneous optimism or pessimism about future return amplifies the technology shock through amplified intertemporal substitution of consumption and leisure, in particular when the belief is reinforced by the data. Their model also generates forecast errors that are consistent with business cycle properties of forecast error for many variables from survey data.

Our paper is also related to a strand of literature explores the role of self-referential learning in explaining asset pricing facts or prices boom and bust. Timmermann (1996) examines the role of learning about stock prices in endowment economy. Carceles-Poveda and Giannitsarou (2008) studies asset pricing with learning in production economy and capital accumulation. Other examples include Lansing (2010), Adam, Marcet, Nicolini (2009) and Adam and Marcet (2010).

3 The Benchmark Model

In this section we present our benchmark model, which is built upon the basic version of the KM model. Then we briefly discuss the steady state and the stochastic rational expectation equilibrium of the model.

3.1 The Model Setup

There are two goods in the economy, a durable asset, land, and a nondurable consumption good, which grows on land but cannot be stored. The durable assets play a dual role, not only as factors of production, but also serve as collateral for loans. There is a continuum of infinitely lived households, with two types, i.e., farmers and gatherers. Both have population size 1 and are risk neutral. Both farmers and gatherers produce and eat consumption good. At each date $t$, there are two markets. One is a competitive
spot market in which land is exchanged for consumption at a price of \( q_t \), the other is a one-period credit market in which one unit of consumption at date \( t \) is exchanged for a claim to \( R_t \) units of consumption at date \( t + 1 \).

The expected utility of a farmer \( i \) is

\[
E_0^{P_i} \sum_{t=0}^{\infty} (\beta^F(i))^t c_t^F(i)
\]

where \( \beta^F(i) \) is subjective discount factor and \( c_t^F(i) \) consumption of the farmer \( i \) at period \( t \). The operator \( E_0^{P_i} \) denotes the farmer \( i \)'s expectation in some probability space \((\Omega, S, \mathcal{P}^i)\), where \( \Omega \) is the space of payoff relevant outcomes that the agent takes as given in its optimization problem. The probability measure \( \mathcal{P}^i \) assigns probabilities to all Borel subsets \( S \) of \( \Omega \). Further details about the \( \Omega \) and \( \mathcal{P}^i \) will be provided after we have described the rest of the economy.

The farmer \( i \) has a constant return to scale technology to produce. The production function is

\[
y_{t+1}^F(i) = (a + a\epsilon_t + \bar{c})K_t^F(i)
\]

where \( K_t^F(i) \) is used land and \( \epsilon_t \) is a stochastic shock to the technology producing tradable good, which is independent and identically distributed as \( N(0, \sigma^2) \). Only \( (a + a\epsilon_t)K_t^F(i) \) of the output is tradable in the market. \( \bar{c}K_t^F(i) \) of output is perishable and nontradable. The introduction of nontradable output is to avoid continually postpone of consumption.

Farmers’ production technology is idiosyncratic in the sense that it requires their specific labor input, which is assumed to be inalienable. \(^5\) The farmers are potentially credit-constrained. Lenders protect themselves against risk of default by collateralizing the farmers’ land. The farmers can at most pledge collateral \( E_t^{P_j} q_{t+1} K_t^F(i) \), and their borrowing constraint is

\[
b_t^F(i) \leq \frac{E_t^{P_j} q_{t+1} K_t^F(i)}{R_t}
\]

where \( b_t^F(i) \) is the amount of loan borrowed by a farmer and \( E_t^{P_j} q_{t+1}(i) \) is the gatherer \( j \)'s expectation of future collateral price. The borrowing constraint says that they can get a loan at most equal to expected current value of his current land holding reselling at \( t + 1 \).

The farmer faces a flow-of-funds constraint

\[
q_t(K_t^F(i) - K_{t-1}^F(i)) + R_t b_{t-1}^F(i) + c_t^F(i) \leq y_t^F(i) + b_t^F(i)
\]

\(^5\)Hart and Moore (1994) show that if part of the value of a project resides in the entrepreneur’s human capital, who is always free to walk away from the project (his human capital is inalienable), then an investor will not be willing to lend more than the liquidation value of the project, even if it was profitable to invest more.
They produce consumption good using land, and borrow from credit market. They spend on consuming, repaying the debt, and investing in land.

A gatherer j’s preference is specified by a linear utility function and she maximizes expected utility

$$E_0^P \sum_{t=0}^{\infty} (\beta^G(j))^t c^G_t(j)$$

where $\beta^G(j)$ is the gatherer j’s subjective discount factor. She faces the following budget constraint:

$$q_t(K^G_t(j) - K^G_{t-1}(j)) + b^G_t(j) + c^G_t(j) \leq y^G_t(j) + R_t b^G_{t-1}(j)$$

where $K^G_t(j) - K^G_{t-1}(j)$ is gather j’s investment in landholding. The production function of the gatherer $j$ is $y^G_{t+1}(j) = G^G_t(K^G_t(j))$ where $G^G' > 0$, $G^G'' < 0$, and $G^G'(0) > aR > G^G''(\bar{K})$.

A few assumptions are made following the KM paper. The supply of land is assumed to be fixed at $\bar{K}$. Later we will assume that all farmers (gatherers) have the same subjective discount factor $\beta^G = \beta^G(j)$ for $\forall j$ ($\beta^F = \beta^F(i)$ for $\forall i$) and farmers are less patient than gatherers $\beta^G > \beta^F$. In addition, assumption $\tilde{c} > (1/\beta^F - 1)a$ is made to ensure that in equilibrium the farmer will not want to consume more than the bruised consumption goods.$^6$

### 3.2 Optimality and Market Clearing Conditions under RE

Given farmers are less impatient than borrowers, in equilibrium, the farmers will borrow from gatherers and the rate of interest rate always equal to the gatherers’ constant rate of time preference; that is $R_t = 1/\beta^G = R$.

Recall how individual farmer $i$ makes her optimal decisions with respect to consumption, borrowing and land demand in original KM paper. Since return to investment is sufficiently high as shown in KM, she prefers to borrow up to the maximum, consume only the nontradable part of his output and invest the rest in landholding. Her optimal consumption is

$$c^F_t(i) = \tilde{c}K^F_{t-1}(i)$$

and optimal borrowing

$$b^F_t(i) = \frac{E^P_t q_{t+1} K^F_t(i)}{R}$$

$^6$The overall return from farming is high enough so the farmers will always use his tradable output for investment. Consider a marginal unit of tradable consumption at date $t$. The borrower could consume it and get utility 1. Alternatively she could invest it in collateral holding and produce consumption goods. In the next period, she will consume the nontradable part of production and invest further the tradable part, and so forth. KM shows that the discounted sum of utility of investing it at date $t$ will exceed utility of immediately consuming it, which is 1. Similarly, the return to investment will also be larger than the other choice that save it for one period and then invest. Hence the collateral constraint will always be binding.
Given that the farmer will consume only the nontradable output, her net worth at the beginning of date $t$ contains the value of her tradable output and land held from previous period, net of debt payment, $Rb_{t-1}^F(i)$. Farmer $i$’s demand on land is

$$K_t^F(i) = \frac{1}{q_t - \frac{1}{R}E_t^{P_j}q_{t+1}}[(a + a\epsilon_t + q_t)K_{t-1}^F(i) - Rb_{t-1}^F(i)]$$

(8)

where $q_t - \frac{1}{R}E_t^{P_j}q_{t+1}$ is the down-payment required to buy a unit of land.

Except for initial period, every period farmer $i$ inherits debt from previous period

$$b_{t-1}^F(i) = \frac{E_{t-1}^{P_j}q_t}{R}K_{t-1}^F(i)$$

(9)

The debt repayment $Rb_{t-1}^F(i)$ is influenced by the time $t$ expectation of collateral price formed at period $t-1$, i.e., $E_{t-1}^{P_j}q_t$, in the case the borrowing constraint is binding. Combining equations (8) and (9), we derive optimal collateral demand of farmer $i$ as following:

$$K_t^F(i) = \frac{1}{q_t - \frac{1}{R}E_t^{P_j}q_{t+1}}[a + \epsilon_t a + q_t - E_{t-1}^{P_j}q_t]K_{t-1}^F(i)$$

(10)

A gatherer $j$ is not credit constrained and her demand for land is determined at the point at which the present value of marginal product of land is equal to the opportunity cost, or user cost of holding land:

$$\frac{1}{R}G_t'(K_t^G(j)) = u_t^c = q_t - \frac{1}{R}E_t^{P_j}q_{t+1}$$

(11)

Market clearing implies $K_t^F + K_t^G = \bar{K}$ and $b_t^F = b_t^G$. Due to zero net supply of bond and land, aggregate consumption $c_t$ will be equal to the aggregate output $y_t$.

### 3.3 The Steady State and the MSV Stochastic Rational Expectation Equilibrium

Assuming homogeneity among all the farmers and all the lenders, symmetric equilibrium requires $K_t^F = \int_0^1 K_t^F(i) = K_t^F(i)$, $K_t^G = \int_0^1 K_t^G(j) = K_t^G(j)$, $b_t^F = \int_0^1 b_t^F(i) = b_t^F(i)$, and $b_t^G = \int_0^1 b_t^G(j) = b_t^G(j)$. There exists a unique non-stochastic steady state. The steady state value of the endogenous variables are $q = \frac{aR}{R-1}$, $u = a$, $K_t^G = G_{t-1}(Ra)$, $K_t^F = \bar{K} - K_t^G$, $b_t^F = qK_t^F/R$ and $c_t^F = c_t^F$.

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7 We assume for initial period (9) also holds.

8 A related paper by Assenza and Beradi (2009, JEDC) enriched Kiyotaki and Moore model with adaptive learning focusing on voluntary default of borrowers. Kuang (2010) shows the “optimality” conditions in AB imply that agents’ “optimal” choices are either suboptimal or infeasible. It also discusses whether this may affect the E-stability condition of the RE equilibrium, propagation of productivity shocks, and the timing of default of borrowers under heterogenous learning rule.
Suppress the index of agents here and let $\frac{1}{\eta}$ denote the elasticity of users’ cost of collateral with respect to borrowers’ collateral holding $\frac{1}{\eta} \equiv \frac{d\log u(P)}{d\log K^F_t} \bigg|_{K^F_t=K^F} = -\frac{d\log G^0(P)(K^G_t)}{d\log K^G_t} \bigg|_{K^G_t=K^G} \times \frac{K^F_t}{K-K^F_t}$. Appendix A shows that log-linearization of the above system (10) and (11) leads to the following equations

$$\hat{q}_t = \gamma_1 E^P_t \hat{q}_{t+1} - \gamma_2 E^P_{t-1} \hat{q}_t + \gamma_3 (\hat{K}^F_{t-1} + \epsilon_t) \quad (12)$$

and

$$\gamma_3 \hat{K}^F_t = \hat{q}_t - \frac{1}{R} E^P_t \hat{q}_{t+1} \quad (13)$$

where $\gamma_1 = \frac{1}{R} (1 + \frac{1}{\eta})$, $\gamma_2 = \frac{1}{\eta}$, and $\gamma_3 = \frac{R-1}{R} \frac{1}{\eta}$.

Denote the parameters with a “$\bar{}$” the value appears in the rational expectation solution. We derive the Minimum State Variables (MSV) stochastic Rational Expectation Equilibrium of our benchmark economy as following:

$$\hat{q}_t = \varphi^m + \varphi^p \hat{K}^F_{t-1} + \varphi^s \epsilon_t \quad (14)$$

$$\hat{K}^F_t = \varphi^m + \varphi^p \hat{K}^F_{t-1} + \varphi^s \epsilon_t \quad (15)$$

where $\varphi^m = 0$, $\varphi^p = \frac{R-1}{R} \frac{1}{\eta+1} \frac{1}{\eta}$, $\varphi^s = \frac{1}{\eta}$, $\varphi^m = 0$, $\varphi^p = \frac{n}{1+\eta}$, and $\varphi^s = \frac{n(1-\frac{1}{R})}{(n+1)(1-\frac{1}{R})}$.

Denote $y_t$ the aggregate output in period $t$, which is

$$y_t = y_t^F + y_t^G \quad (16)$$

$$= (a + \bar{\tau}) K^F_{t-1} + G(K^G_{t-1}) \quad (17)$$

Denote $Y$ the steady state value of aggregate output. Log-linearization of aggregate output (17) will yield

$$\hat{y}_t = \frac{(a + \bar{\tau}) - G' (a + \bar{\tau}) K^F_{t-1} + G(K^G_{t-1})}{Y} \hat{K}^F_{t-1}$$

Output will be equal to the product of productivity gap $\frac{(a + \bar{\tau}) - G'}{(a + \bar{\tau})}$, production share of borrowers $\frac{(a + \bar{\tau}) K^F_{t-1}}{Y}$ and the redistribution of collateral, so does aggregate consumption $\hat{c}_t$.

### 4 Equilibrium with Imperfect Knowledge

We still assume homogeneous expectation among all agents but relax the assumption that the homogeneity of the agents is common knowledge, in particular, the agents do not know other agents’ discount factor and belief about future collateral price.\(^9\)\(^10\)

\(^9\)Since the agents are risk neutral, the equilibrium interest rate is constant every period and equal to $\frac{1}{\rho}$ and the agents do not need to learn about interest rate.

\(^10\)We assume the agents are not concerned about other agents’ model or belief. They believe their model is the best way to form expectation and there is no higher order belief in the learning model.
We firstly discuss the underlying probability space conditional on which agents form their expectation and the equilibrium concept of our model. Afterwards we specify agents’ near REE belief and study their optimal learning behavior given their belief and information set.

### 4.1 The Underlying Probability Space and the Internal Rational Expectation Equilibrium

We assume agents’ belief system has the same form as the expression for the collateral price process in the Stochastic Rational Expectation Equilibrium (14) and (15). Agents believe collateral prices and borrowers’ collateral holding depend on past aggregate borrowers’ collateral holding. We now describe the probability space \((\Omega, S, \mathcal{P})\). Both borrowers and lenders view the process for \(q_t, \epsilon_t\) and \(K_F^t\) as external to their decision problem and the probability space over which they condition their choices is given by \(\Omega = \Omega_q \times \Omega_\epsilon \times \Omega_{K^F}\) where \(\Omega_X = \prod_{t=0}^{\infty} R_+\) and \(X \in \{q, \epsilon, K^F\}\). The probability spaces contain all possible sequences of prices, productivity shock and borrowers’ collateral holding. We denote the set of all possible histories up to period \(t\) by \(\Omega^t\) and its typical element is denoted by \(\omega^t \in \Omega^t\).

The agents are assumed to be “Internal Rational”\(^{11}\) as defined below, i.e., maximize their expected utility under uncertainty, taking into account all their constraints, and condition their choices variables over the history of all external variables. Their expectation about future external variables are evaluated based on their consistent set of subjective beliefs specified in the subsequent subsection, which is endowed to them at the outset.

**Definition 1 Internal Rationality**

a) A farmer \(i\) is “Internal Rational” if she chooses \((b_i^F(i), K_i^F(i), c_i^F(i)) : \Omega^t \to R^3\) to maximize expected utility (1) subject to the flow-of-fund constraint (3), the collateral constraint (2) and her production function, taking as given the probability measure \(\mathcal{P}^i\).

b) A gatherer \(j\) is “Internal Rational” if she chooses \((b_j^G(j), K_j^G(j), c_j^G(j)) : \Omega^t \to R^3\) to maximize expected utility (4) subject to the flow-of-fund constraint (5) and her production function, taking as given the probability measure \(\mathcal{P}^j\).

Note internal rationality of the agents are not tied to any specific belief system or to learning behavior of the agents. However, the belief system are usually specified with some near-rationality concept and it is natural to introduce learning behavior of the agents.

In the following we specify the equilibrium of the economy. Let \((\Omega_\epsilon, S_\epsilon, P_\epsilon)\) be a probability space over the space of histories of productivity shocks \(\Omega_\epsilon\) and \(P_\epsilon\) denoting

\(^{11}\)This follows Adam and Marcet (2011).
the 'objective' probability measure for productivity shock. Let $\omega \in \Omega$ denote a typical infinite history of productivity shock.

**Definition 2** Internal Rational Expectation Equilibrium

The Internal Rational Expectation Equilibrium (IREE) consists of a sequence of equilibrium price functions $\{q_t\}_{t=0}^{\infty}$ where $q_t : \Omega^t \to \mathbb{R}_+$, contingent choices $(c_t^F(i), c_t^C(j), b_t^F(i), b_t^C(j), K_t^F(i), K_t^C(j)) : \Omega^t \to \mathbb{R}^6$ and probability belief $\mathcal{P}^i$ for each farmer $i$ and $\mathcal{P}^j$ for each gatherer $j$, such that

1. All agents are internal rational, and
2. when agents evaluate $(c_t^F(i), c_t^C(j), b_t^F(i), b_t^C(j), K_t^F(i), K_t^C(j))$ at equilibrium prices, markets clear for all $t$ and all $\omega \in \Omega$ almost surely in $P$.

In the Internal Rational Expectation Equilibrium, expectations about collateral prices are formed based on agents' subjective belief system, which is allowed not necessarily equal to the objective density. Collateral prices and collateral holdings are determined by equation (12), and (13) after we specify agents' probability measure $\mathcal{P}$.

### 4.2 Agents' Belief System and Optimal Learning Behavior

We now describe agents' probability measure $\mathcal{P}$ and derive their optimal learning rule. We assume agents’ subjective belief system is such that they believe collateral prices and current collateral holding depend on past collateral holding\(^\text{12}\) as under rational expectation. It could be represented as following:

\[
\begin{align*}
\hat{q}_t &= \phi^m + \phi^p \hat{K}_t^{F-1} + \omega_t \\
\hat{K}_t^F &= \chi^m + \chi^p \hat{K}_t^{F-1} + \theta_t
\end{align*}
\]

given $\hat{K}_t^{F-1}$ and

\[
\begin{pmatrix}
\omega_t \\
\theta_t
\end{pmatrix} \sim \text{i.i.d. } N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma^2_{\omega} & 0 \\
0 & \sigma^2_{\theta}\end{pmatrix}\right)
\]

However, they are assumed to be uncertain about the parameters and shock precisions $(\phi^m, \phi^p, \sigma^2_{\omega}, \chi^m, \chi^p, \frac{1}{\sigma^2_{\theta}})$. Optimal behavior implies that agents' beliefs are updated by applying Bayes’ law to market outcomes. Since uncertainty and learning about $(\chi^m, \chi^p, \frac{1}{\sigma^2_{\theta}})$ does not matter for agents’ one-step ahead forecast of collateral prices, we omit the belief updating equation for $(\chi^m, \chi^p, \frac{1}{\sigma^2_{\theta}})$ in the rest of the paper.

\(^{12}\)This is analogous to learn the parameter linking prices and dividend. Note the dividend here is marginal product of lender or user's cost and function of borrowers' collateral holding. After log-linearization, (percentage deviation of) dividend is just a constant multiple of (percentage deviation of) borrowers' collateral holding.
Denote $\phi \equiv (\phi^m, \phi^p)$ the intercept and slope in agents’ belief system. Their uncertainty are summarized by a distribution

$$(\phi^m, \phi^p, H) \sim f$$

The prior distribution of unknown parameters is assumed to be a Normal-Gamma distribution as following

$$H_0 \sim G(\gamma_0, s_0^{-2})$$

$$(\phi_0^m, \phi_0^p)' | H_0 \sim N((\theta_0^m, \theta_0^p)', (\nu_0 H)^{-1})$$

The precision is distributed as a Gamma distribution, and conditional on the residual precision the unknown parameters are jointly normally distributed. The deviation of our prior with the REE prior will vanish assuming agents’ initial belief is at the RE value $\theta = \bar{\theta} = (\bar{\theta}^m, \bar{\theta}^p)'$, they have infinite confidence on their belief about the parameters, i.e., $\gamma_0 \to \infty$, and $\nu_0 \to \infty$.

For the sake of notation compactness, for the rest of this section we denote $y_t$ the dependent variable $q_t$ in agents’ regression, and $x_t$ regressors, which correspond to $(1, \bar{R}_{t-1}^F)$. $\theta \equiv (\theta^m, \theta^p)$ stands for the posterior mean of $(\phi^m, \phi^p)$.

Given agents’ prior belief (21) and (22), they learn unknown parameters in light of data using Bayes’ law. Appendix B shows the posterior distribution of unknown parameters is given by

$$H_t|\omega^t \sim G(\gamma_t, s_t^{-2})$$

$$(\phi_t^m, \phi_t^p)'|H_t = h, \omega^t \sim N((\theta_t^m, \theta_t^p)', (\nu_t h)^{-1})$$

where the parameters $(\theta_t^m, \theta_t^p, \nu_t, \gamma_t, s_t^{-2})$ evolve recursively as following:

$$\begin{align*}
\theta_t &= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \\
\nu_t &= \nu_{t-1} + x_t x_t' \\
\gamma_t &= \gamma_{t-1} + \frac{1}{2} \\
s_t^{-2} &= s_{t-1}^{-2} + \frac{1}{2} (y_t - x_t' \theta_{t-1})'(x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} (y_t - x_t' \theta_{t-1})
\end{align*}$$

To avoid simultaneity between agents’ belief and actual outcomes, we assuming information on data is introduced with a delay in $\theta_t$. So that we actually use

$$\begin{align*}
\theta_t &= \theta_{t-1} + (x_{t-1} x_{t-1}' + \nu_{t-1})^{-1} x_{t-1} (y_{t-1} - x_{t-1}' \theta_{t-1}) \\
\nu_t &= \nu_{t-1} + x_{t-1} x_{t-1}'
\end{align*}$$

A microfounded belief system justifying this delay could be provided following Adam and Marcet (2010).
Equations (29) and (30) are equivalent to the following Recursive Least Square (RLS) learning algorithm

$$\theta_t = \theta_{t-1} + \frac{1}{t+N} R_{t-1}^{-1} x_{t-1}(y_{t-1} - x_{t-1}^{t-1} \theta_{t-1})$$

$$R_t = R_{t-1} + \frac{1}{t+N} (x_{t-1} x_{t-1}^{t-1} - R_{t-1})$$

when we set initial parameters $\nu_0 = NR_0$. Then it could easily be shown that for subsequent periods we have $\nu_t = (t + N)R_t$, for $\forall t \geq 1$. Therefore, $N$ in above equations could be interpreted as measure of the precision of initial belief.

5 Understanding the Learning Model

We firstly provide some preliminary view on why learning model amplifies and propagates the shock relative to a RE version of the model. Then we analyze the learning dynamics in further details. We investigate the E-stability of the RE equilibrium. In addition, we show our learning model with collateral constraint may generate momentum in beliefs and collateral prices. Finally we analyzes the dependence of momentum in beliefs on key parameters.

5.1 Preliminary View on the Mechanism

For comparison of different dynamics under RE and learning, we consider a one-time unanticipated shock, for example, a positive productivity shock to borrowers’ technology producing tradable output. We assume the economy starts at its non-stochastic steady state. Log-linearization of borrowers’ collateral demand equation (10) yields

$$\hat{K}^F_t = \frac{R}{R-1} [(\hat{q}_t - E_t^{sp} \hat{q}_t) - (\hat{q}_t - \frac{1}{R} E_t^{sp} \hat{q}_{t+1})] + \hat{K}^F_{t-1} + \epsilon_t$$

The dynamics under rational expectation is analyzed in the original KM paper. Borrowers’ demand on collateral increases after positive productivity shock. In the impact period, collateral is transferred from lenders to borrowers. Due to fixed supply of land and that lenders has decreasing return to scale technology to produce, the users’ cost of collateral rises above its steady state value. Also the intertemporal investments in collateral holding are complement to each other. In particular, borrowers today’s investment in collateral holding raises tomorrow’s ability to borrow, there will be persistence in agents’ collateral holding. The users’ cost of collateral stays above the steady state for some periods. Under RE, collateral prices are discounted sum of current and future users’ cost. Persistence in users’ cost reinforces the effect on collateral prices and collateral value, which leads to larger effect on collateral transfer and
aggregate activities. After the shock disappears, expectations about future collateral prices realize themselves and there will be no capital gain or loss in collateral holding. Higher than steady state value of users’ cost chokes off further rise in borrowers’ demand on collateral. Collateral prices and borrowers’ collateral holding will revert immediately toward the steady state. The prices and quantities converge persistently and monotonically to the steady state.

Now we turn to the dynamics under the learning model. Assume initially agents’ belief about unknown parameters is at the RE level. So the learning agents have correct forecast function initially. The impact responses of all variables are the same as that under Rational Expectation. However, after the shock disappears, unlike under RE, every period agents’ expectation about future collateral price may not be realized during learning transition. Borrowers may make persistent expectation error, which induces capital gain or loss in their collateral holding and additional fluctuation in their net worth.

Due to learning about parameters in the collateral prices process, agents interpret partially the surprise in collateral prices or expectation errors, which might be temporary fluctuation, as permanent changes in parameters of this process. They adjust their belief, which generates more optimistic expectation about future collateral prices than under RE. Collateral constraint is relaxed based on lenders’ optimistic expectation about the liquidation value of collateral. With larger borrowing capacity, borrowers could have larger demand on collateral, which boosts collateral prices further up. From equation (31), agents will increase their demand for collateral as long as capital gain outweighs the users’ cost of collateral. Learning about collateral prices give rise to dynamic feedback among agents’ belief, credit limit and actual prices, which generates additional propagation of the shock as well as prolonged periods of expansion of prices and quantities. We will analyze later in more details about a key property of the model dynamics: momentum in beliefs, and its possible dependence on key parameters: leverage ratio and the elasticity of users’ cost with respect to borrowers’ collateral holding $\frac{1}{q}$.\(^{13}\)

Collateral price booms could be choked off for a number of reasons. For example,\(^ {13}\)A limiting case is that the elasticity of users’ cost with respect to borrowers’ collateral holding $\frac{1}{q} \to 0$, which amounts to the lenders’ marginal product is constant $G'$. The equations (10) and (11) become

\[
K_t^F = \frac{1}{qt - \frac{1}{R_t} E_{t+1}^P q_{t+1}} (qt - E_{t-1}^P q_t) K_{t-1}^F
\]

\[
= \frac{1}{R_t G'} (a + qt - E_{t-1}^P q_t) K_{t-1}^F
\]

\[
\frac{1}{R_t} G' = u_t^e = qt - \frac{1}{R_t} E_{t}^P q_{t+1}
\]

Learning about collateral prices generating fluctuation of borrowers’ net worth but transfer of collateral between borrowers and lenders does not affect lenders’ marginal product or users’ cost and has no effect on the collateral prices. Note in this case collateral prices will not be affected by borrowers’ productivity shock but will be influenced by other types of shock such as shock to lenders’ productivity
some adverse fundamental shocks such as shock to interest rate or productivity, or the endogenous model dynamics could lead to that the capital gain is smaller than the rising users’ cost. Borrowers will then start to reduce their demand on collateral, and collateral prices revert subsequently. When collateral prices fall short of agents’ expectation, their beliefs are updated downward and agents become pessimistic. The realization of collateral prices implied by the actual law of motion will be low, which leads to further pessimism of agents. The dynamic interaction among belief, credit limit and realizations induce faster decline in price and quantities.

The previous adaptive learning literatures find that learning the parameter linking price and dividend/capital do not generate sufficiently large effect on asset prices volatilities comparable to the data in an endowment economy or production economy without collateral constraint. Even though agents learn such a link, the learning effect could be large in our model due to collateral constraint and momentum. Our model with collateral constraint has richer asset pricing equation. The inherited debt holding is influenced by agents’ expectation about current collateral prices formed previously given that we consider only non state-contingent debt in this paper. The debt holding will in turn affect agents’ net worth and hence on current collateral prices. Asset prices in our model depend not only on current belief but also on change of beliefs.

5.2 Belief Dynamics

We now analyze in more details the belief dynamics of our learning model. We investigate whether and under which conditions (i.e., the Expectational-Stability condition) agents’ belief will converge to the REE beliefs by using stochastic reversive algorithm (SRA) techniques elaborated in Evans and Honkahpohja (2001). Furthermore, we examine an interesting property of our model during learning transition.

Agents believe collateral prices depend on collateral holding, i.e., they use collateral holding to forecast future collateral prices. The state variables of the learning algorithm are $x_t = (1 \ K^F_t)'$. Agents’ conditional expectations are $E^P_t \tilde{q}_t = \phi_{t-1} x_{t-1}$ and $E^P_t \tilde{p}_{t+1} = \phi_t x_t$ where $\phi_t \equiv (\phi_t^m, \phi_t^p)'$. Substituting the conditional expectations into the model equations (12) and (13), we arrive at the actual law of motion for collateral prices of our learning model

\begin{align*}
\tilde{q}_t &= T_1(\phi_{t-1}^m, \phi_t^m, \phi_t^p) + T_2(\phi_{t-1}^p, \phi_t^p) \tilde{K}^F_{t-1} + T_3(\phi_t^p) \epsilon_t \\
\tilde{K}^F_t &= \frac{\tilde{q}_t - \phi_t^m / \gamma_3}{\gamma_3 + 1 / \phi_t^p}
\end{align*}

where $T_1(\phi_{t-1}^m, \phi_t^m, \phi_t^p) = \frac{(\gamma_1 + \gamma_2 \phi_{t-1}^m) (1 + \phi_t^p / \gamma_3) - \gamma_1 \phi_t^m \phi_t^p}{1 + \phi_t^p / \gamma_3 - \gamma_1 \phi_t^m / \gamma_3}$, $T_2(\phi_{t-1}^p, \phi_t^p) = \frac{\gamma_3 - \gamma_2 \phi_t^p / \gamma_3}{1 - \gamma_1 \phi_t^m / \gamma_3}$ and $T_3(\phi_t^p) = \frac{\gamma_3}{1 - \gamma_3 \phi_t^p / \gamma_3}$. Agents’ beliefs about the unknown parameters $\phi$ are optimally

or to interest rate.
updated by the following RLS learning rule

\[ \phi_t = \phi_{t-1} + \frac{1}{t+N} R_{t-1}^{-1} x_{t-1} (\hat{q}_{t-1} - \phi'_{t-1} x_{t-1}) \]  

(34)

\[ R_t = R_{t-1} + \frac{1}{t+N} (x_{t-1} x'_t - R_{t-1}) \]  

(35)

We define \( \Theta_0 \) the set of admissible parameters in our benchmark learning model.

**Definition 3**

The set of admissible parameters, \( \Theta_0 \equiv \{(\eta, R)|\eta > 0, R > 1\} \).

The T-map mapping from agents’ PLM to ALM is \( T(\phi^m, \phi^p) \equiv (T_1,T_2)(\phi^m, \phi^p) \equiv \left( \frac{\gamma_1 \phi^m - \gamma_2 \phi^m (1+\phi^p R) - \frac{\gamma_1}{\gamma_3} \phi^m R}{1+\frac{\gamma_1}{\gamma_3} \phi^m R - \frac{\gamma_2}{\gamma_3} \phi^m R}, \frac{\gamma_3 - \gamma_2 \phi^p}{1 - \frac{\gamma_1}{\gamma_3} \phi^m R} \right) \). Local stability of the MSV REE is determined by the stability of the following associated ODEs

\[ \frac{d\phi^m}{d\tau} = T_1(\phi^m, \phi^p) - \phi^m \]

\[ \frac{d\phi^p}{d\tau} = T_2(\phi^p, \phi^p) - \phi^p \]

The following condition establishes a sufficient condition for E-stability of the MSV equilibrium (14).

**Proposition 4**

The MSV equilibrium (14) for the economy represented by Eqs. (12), and (13) is E-stable for any admissible parameters in \( \Theta_0 \).

**Proof.** See Appendix C. \( \blacksquare \)

The users’ cost of collateral plays important role in stabilizing collateral demand and prices around the neighborhood of the REE equilibrium. This could be seen more clearly after we reformulate equation (31) as

\[ \hat{K}_t^F = \frac{R}{R-1} \left[ \frac{1}{R} E^P_t \hat{q}_{t+1} - E^P_t \hat{q}_t \right] + \hat{K}_{t-1}^F \]  

(36)

The following illustration may help to understand the E-stability condition. Fix agents’ belief at \( \phi^m \) is at RE value 0 and \( \phi^p \) above RE value, which implies that there is small deviation of collateral price expectation above and away from rational expectation level. The conditional expectations \( E^P_t \hat{q}_{t+1} = \phi^p \hat{K}_{t-1}^F \) and \( E^P_t \hat{q}_{t+1} = \phi^p \hat{K}_{t-1}^F = \phi^p \frac{\hat{q}_t}{\gamma_3 + \frac{1}{R} \phi^p} = \frac{\phi^p}{\gamma_3 + \frac{1}{R} \phi^p} T_2(\phi^p, \phi^p) \hat{K}_{t-1}^F \). The actual elasticity of collateral prices with respect to collateral holding \( T_2(\phi^p, \phi^p) \) is low such that for all admissible parameters \( \frac{\phi^p}{\gamma_3 + \frac{1}{R} \phi^p} T_2 < \phi^p \). The
users’ cost of collateral outweigh capital gain due to \( \frac{1}{R} E_t^P \tilde{q}_{t+1} < E_t^P \tilde{q}_t \), which pushes the collateral holding and prices back to the RE equilibrium. Therefore, the asymptotic local stability of the REE is achieved. Roughly speaking, given that the E-stability condition is satisfied and the estimates are around the neighborhood of steady state, we have \( \beta_t \to \beta \) and \( \nu_t \to \infty \) almost surely.

Although the price and expectation dynamics generated under this belief system converge over time to the rational expectations, this process takes a long time to converge. Along the learning transition, prices and expectation dynamics oscillate for a long time around the rational expectation equilibrium. Agents’ belief may display momentum, which is defined formally below. The discussion of its economic implications is delayed in a few lines.

**Definition 5 Momentum**

Denote \( b_t \) agents’ belief (parameter estimate) at period \( t \), and \( \bar{b} \) the corresponding value at the RE level. Momentum is defined as:

1. if \( b_t \leq \bar{b} \) and \( b_t > b_{t-1} \), then \( b_{t+1} > b_t \).
2. if \( b_t \geq \bar{b} \) and \( b_t < b_{t-1} \), then \( b_{t+1} < b_t \).

We discuss the deterministic dynamics of the learning model assuming \( \epsilon_t = 0 \) for all \( t \). In particular, the learning economy may display momentum in agents’ belief described above. A simplified PLM without learning about \( \phi^m \) or the steady state, that is, \( \tilde{q}_t = \phi^P_{t-1} \tilde{K}^F_{t-1} + \omega_t \) is considered. We focus on properties of the T-map mapping perceived persistence to the actual persistence, \( T_2(\phi^P_{t-1}, \phi^P_t) = \frac{\gamma_3 - \gamma_2 \phi^P_{t-1}}{1 - \gamma_3 + \phi^P_t} \), which also determine critically the dynamics of model with learning about \( \phi^m \).

Even if agents learn the parameter linking (log deviation of) collateral prices and collateral holding, the T-map, \( T_2(\phi^P_{t-1}, \phi^P_t) = \frac{\gamma_3 - \gamma_2 \phi^P_{t-1}}{1 - \gamma_3 + \phi^P_t} \), depends both the level and change of agents’ belief. Current collateral prices depend not only on current belief about future prices but also on inherited debt holdings through agents’ net worth. The debt holdings are in turn influenced by expectation of current collateral prices formed at previous period. So past belief about persistence comes into play.

It is possible the following dynamics exists in the learning model. When agents’ belief arrived at rational expectation level from below (above), that is \( \phi^P_{t-1} < \phi^P_t < \bar{\phi} \) \((\phi^P_{t-1} > \phi^P_t > \bar{\phi})\), the realization of parameter in actual law of motion \( T_2(\phi^P_{t-1}, \phi^P_t) \) implied by the T-map could be higher (lower) than the RE value \( \bar{\phi} \). Using realized collateral prices, agents update their belief further upward (downward), i.e., \( \phi^P_{t+1} > \phi^P_t \) \((\phi^P_{t+1} < \phi^P_t)\). Momentum in beliefs helps generating large volatility in prices, return and quantities.

\(^{14}\)This follows Adam, Marcet and Nicolini (2009).
We look into whether learning model with this belief system could display momentum in collateral prices. The following result shows that momentum in beliefs is more easily arises in learning economy with a higher elasticity of users’ cost of collateral with respect to borrowers’ collateral holding \( \frac{1}{\eta} \) or higher steady state leverage ratio.

**Proposition 6**

A sufficient condition ensuring that the learning economy displays momentum in agents’ belief (around the neighborhood of REE beliefs\(^{15} \)) is either

(1) when \( \frac{1}{\eta} > \frac{1}{3} \),

or

(2) when \( \frac{1}{\eta} \leq \frac{1}{3} \) and the steady state leverage ratio \( 1/R > \frac{1}{R(\eta)} \) with \( R(\eta) \equiv \frac{\eta}{2}[1 - \sqrt{1 - \frac{4}{\eta+1}}] \).

**Proof.** See Appendix D. \( \blacksquare \)

In figure 3, we plot the threshold leverage ratio (or loan-to-value ratio) \( \frac{1}{R} \) as function of \( \frac{1}{\eta} \), i.e., \( \frac{1}{R}(\frac{1}{\eta}) \), and summarize parameter combinations that generate momentum in beliefs in the shaded area. The threshold leverage is an decreasing function in \( \frac{1}{\eta} \) if \( \frac{1}{\eta} < \frac{1}{3} \). This proposition says regardless of the steady state leverage ratio, the learning economy exhibits momentum in beliefs if the elasticity of users’ cost with respect to borrowers’ collateral holding is larger than \( \frac{1}{3} \). When the elasticity \( \frac{1}{\eta} \) is relatively small, the model exhibits self-reinforcing dynamics only when leverage ratio is sufficiently high. It can be seen from figure 3, for relative small \( \frac{1}{\eta} \), momentum in beliefs could be present in a learning economy with higher leverage ratio but not with lower leverage ratio.

Recall agents’ current expectation about future collateral prices is \( E_t^P \hat{q}_{t+1} = \phi_t^P \hat{K}_t^F \). From equation 31, with more optimistic expectation about future prices it is more easily that the users’ cost of collateral stays below the capital gain of collateral holding given agents’ expectation formed previously, i.e., \( E_t^P \hat{q}_t \). The larger the elasticity \( \frac{1}{\eta} \) is, the larger is the increase in collateral prices in response to a positive shock to collateral prices, e.g., a productivity shock and so does borrowers’ collateral holding. So the expectation about future collateral price will be more optimistic with a larger \( \frac{1}{\eta} \). Alternatively, with larger steady state leverage ratio or more effectiveness of collateral, response of prices and quantities will be amplified more and expectation will be more optimistic.

Note we consider gross interest rate is not too high, say, \( R < 1.5 \), otherwise agents would prefer save to invest in collateral holding, as indicated by assumption 2 of the KM paper, i.e., \( \bar{c} > (\frac{1}{p^{\ast}} - 1)a \). In the benchmark model, the gross interest rate \( R \) determines the steady state leverage ratio \( \frac{1}{R} \), so in figure 3 the steady state leverage ratio is above \( \frac{2}{3} \). Below we extend the model such that steady state leverage ratio is separated from the interest rate.

\(^{15}\)Due to the denominator is nonlinear in current belief \( \phi_t^P \), a first-order Taylor expansion of the denominator around the REE belief is done for deriving this proposition. See Appendix H.
Figure 3: Threshold function $\frac{1}{R}(\frac{1}{\eta})$ and parameter combinations generating momentum

6 Model Extension and Modification of Agents’ Belief

We extend our benchmark model with proportional transaction cost $\tau$, so the maximal loan borrowers could get is certain fraction of the collateral value. One explanation is that debt enforcement procedures in real world are significantly inefficient and some value is lost during such procedure, as documented by Djankov, Hart, Mcliesh and Shleifer (2008). In addition, we modify agents’ belief system such that they may perceive the parameters in their subjective model are drifting over time, e.g., random walk. The subsequent quantitative results are based on the modified version of model discussed here.

6.1 Extended Model with Proportional Transaction Cost

We assume that if borrowers repudiate their debt obligations, the lenders can repossess the borrowers’ assets by paying a proportional transaction cost $\tau E^P_t q_{t+1} K^F_t$. The borrower’s collateral constraint now becomes

$$b_t^F \leq (1 - \tau) \frac{E^P_t q_{t+1}}{R} K^F_t$$

(37)

The maximal amount of loan she can get is $(1 - \tau) E^P_t q_{t+1} K^F_t / R$. 
The system of equations representing the dynamics for the extended model economy are following

\[
K_t^F = a + a\epsilon_t + q_t - (1 - \tau)E_t^p q_{t-1}q_t K_t^{F-1}
\]

(38)

\[
K_t^G = G'(Ru_t^s)
\]

(39)

where \( u_t^s = q_t - \frac{1}{R}E_t^p q_{t+1} \). The downpayment for buying one unit collateral, \( q_t - \frac{1}{R}(1 - \tau)E_t^p q_{t+1} \), differs with the users’ cost and the former is now larger. Note the benchmark model is nested in the extended one as a special case when \( \tau = 0 \).

The steady state of the extended model differs with that of the benchmark model, which has \( u = \frac{a}{1-\tau}, q = \frac{aR}{(1-\tau)(1-\tau)}, K^G = G'(\frac{aR}{1-\tau}) \) and \( K^F = \hat{K} - K^G \). The steady state leverage ratio is now \( \frac{1}{R_{\tau}} \).

Log-linearization of borrowers’ collateral demand equation leads to following equation

\[
\hat{q}_t = \xi_1 E_t^p \hat{q}_{t-1} - \xi_2 E_t^p \hat{q}_t + \xi_3 \hat{K}_{t-1}^{F-1} + \xi_4 \epsilon_t
\]

(40)

where \( \frac{\xi_1}{\gamma} = \frac{1}{R} + \frac{1}{\gamma} \frac{1}{R-1} \), \( \xi_2 = (1 - \tau) \frac{1}{R-1} \frac{1}{R} \), \( \xi_3 = \frac{1}{R-1} \frac{1}{R} \), and \( \xi_4 = \frac{\xi_3 R}{\xi_1 (1-\tau)} \). Lenders’ collateral demand equation is unchanged and still equation (13).

The REE collateral price process could be derived as

\[
\hat{q}_t = \zeta^m + \zeta^p \hat{K}_{t-1}^{F-1} + \zeta^s \epsilon_t
\]

(41)

\( \zeta^m = 0, \zeta^p = \frac{1}{\gamma + (1-\tau)(\frac{1}{\gamma} + 1)} \) and \( \zeta^s = \frac{1}{\gamma} \frac{1}{1-\gamma R} \).

Learning agents use collateral holding to forecast collateral prices, so their conditional expectations are \( E_{t-1} \hat{q}_t = \zeta^m_{t-1} + \zeta^p_{t-1} \hat{K}_{t-1}^{F-1} \) and \( E_t \hat{q}_{t+1} = \zeta^m_t + \zeta^p_t \hat{K}_{t}^{F} \). Plugging the expectations into equation (40) and (13), we obtain the actual law of motion for collateral prices

\[
\hat{q}_t = T_1(\zeta^m_{t-1}, \zeta^m_t, \zeta^p_t) + T_2(\zeta^p_{t-1}, \zeta^p_t) \hat{K}_{t-1}^{F-1} + T_3(\zeta^p_t) \epsilon_t
\]

(42)

where \( T_1(\zeta^m_{t-1}, \zeta^m_t, \zeta^p_t) = \frac{\xi_1 \zeta^m_{t-1} - \xi_2 \zeta^m_t (1 + \frac{\zeta^p_t}{\zeta^p_t}) - \xi_3 \zeta^p_{t-1}}{1 + \frac{\epsilon^p_t}{\xi_3 \epsilon^p_t} - \xi_1 \zeta^m_t \frac{1}{\xi_3}} \), \( T_2(\zeta^p_{t-1}, \zeta^p_t) = \frac{\xi_3 - \xi_2 \epsilon^p_{t-1}}{1 - \frac{\xi_1 \epsilon^p_t}{\xi_3 + \xi^p_t \frac{1}{R}}} \), and \( T_3(\zeta^p_t) = \frac{\xi_4 \zeta^p_t}{1 - \frac{\xi_1 \epsilon^p_t}{\xi_3 + \xi^p_t \frac{1}{R}}} \).

Define the admissible parameter space \( \Theta_1 \) as following.

**Definition 7**

The admissible parameter space \( \Theta_1 = \{ (\eta, R, \tau) | \eta > 0, R > 1, 0 \leq \tau < 1 \} \).

The following proposition examines the \( E \)-stability condition of above MSV equilibrium.
Proposition 8

The MSV equilibrium (41) for the economy represented by (40) and (13) is E-stable for all admissible parameter in $\Theta_1$.

Proof. see appendix E. ■

The deterministic dynamics of the learning model is discussed by assuming $\epsilon_t = 0$ for all $t$. A simplified PLM without learning about $\bar{\phi}^m$ or the steady state, that is, $\hat{q}_t = \zeta_{t-1}^p \hat{R}_{t-1}^F + \omega_t$ is considered. We focus on properties of the T-map mapping perceived persistence of the actual persistence, $T_2(\phi_{t-1}^p, \phi_t^p) = \frac{\xi_3 - \xi_4^p \xi_{t-1}^p}{1 - \xi_3 + \xi_4^p \xi_{t-1}^p}$. Again the current collateral prices depend not only on current belief but past belief about current prices, so momentum in beliefs may emerge. The following condition summarizes the dependence of momentum in beliefs on key parameters in the extended learning model.

Proposition 9

A sufficient condition\textsuperscript{16} guarantees momentum in beliefs (around the neighborhood of REE beliefs) is that parameter combinations of $(\eta, R, \tau)$ satisfy $\frac{R - 1}{R} > \frac{1}{g(R)^{\frac{1}{2}} + 1}$ where $g(R) = R(\sqrt{(R - 1) + \frac{(R - 1)^2}{4}} + R - 1)$.\textsuperscript{17}

Proof. see appendix F. ■

As an example, we set gross quarterly rate $R$ to 1.01. Shaded area of figure 4\textsuperscript{17} summarizes the parameter combinations $(\frac{1 - \tau}{R}, \frac{1}{\eta})$ with which there is momentum in beliefs in extended learning model. The threshold steady state loan-to-value ratio as function of $\frac{1}{\eta}$, i.e., $\frac{1 - \tau}{R} = \frac{1}{g(R)^{\frac{1}{2}} + 1}$ is also plotted, which is decreasing in the elasticity $\frac{1}{\eta}$. We can see in the extended learning model, momentum could arise when the elasticity of users’ cost with respect to borrowers’ collateral holding or the steady state leverage ratio is relatively large.

6.2 Modification of Agents’ Belief System

The belief we assumed in section 4.2 implies that agents’ beliefs converge to the REE beliefs and the volatility of prices decreases over time. Below we modify agents’ belief such that they may perceive fundamental parameters $\phi_t$ keeps changing over time. Specifically, agents perceive the following random walk model of coefficient variation

\begin{align*}
\phi_t &= \phi_{t-1} + \epsilon_t & E\epsilon_t \epsilon_t' &= R_{11} \\
y_t &= \phi_t \zeta_t + \xi_t & E\xi_t \xi_t' &= R_{22}
\end{align*}

\textsuperscript{16}It could be shown that when $\tau = 0$, the condition will collapse to the condition in proposition 6.

\textsuperscript{17}The parameter combinations generating momentum in beliefs are not sensitive to the steady state value of interest rate $\hat{R}$.
Figure 4: Threshold function \( \frac{1}{\eta} \left( \frac{1}{\eta} \right) \) and parameter combinations generating momentum

The definition of \( y_t \) and \( x_t \) are the same as in section 4.2. Define \( P_t = E[(\theta_t - \phi_t)(\theta_t - \phi_t)'] \), where \( \theta_t \) stands for agents' estimate for \( \phi_t \). The prior distribution about \( \phi_0 \) is assumed to be normal, i.e., \( N(\theta_0, P_{0|0}) \).

Agents learn unknown parameters via Bayes' law. The posterior mean could be represented by the following basic Kalman filter recursions\(^{18}\)

\[
\begin{align*}
\theta_t &= \theta_{t-1} + L_t [y_t - x_t' \hat{\theta}_{t-1}] \\
L_t &= \frac{P_{t|t-1} x_t}{R_{2t} + x_t' P_{t|t-1} x_t} \\
P_{t+1|t} &= P_{t|t-1} - \frac{P_{t|t-1} x_t x_t' P_{t|t-1}}{R_{2t} + x_t' P_{t|t-1} x_t} + R_{1t+1}
\end{align*}
\]

Derivations of the above Kalman filter could be found in, e.g., Harvey (1989).

Assume further that agents perceive that \( R_{1t} = \frac{g}{1-g} P_{t-1|t-1} \) and \( R_{2t} = \frac{1}{g} \). The above Kalman filter leads to

\[
\theta_t = \theta_{t-1} + L_t [y_t - x_t' \hat{\theta}_{t-1}]
\]

\(^{18}\)Note for the model considered here the prior distribution about \( \phi_t \) is the same as the posterior about \( \phi_{t-1} \), i.e., \( \theta_{t|t-1} = \theta_{t-1|t-1} \), so we suppress the conditioned information set and use \( \theta_t \) for both.
where

\[ L_t = \frac{P_{t-1|t-1} x_t}{1 - g} + x'_t P_{t-1|t-1} x_t \]

\[ P_{t|t} = \frac{1}{1 - g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1} x_t x'_t P_{t-1|t-1}}{x'_t P_{t-1|t-1} x_t + \frac{1-g}{g}} \right] \]

With \( P_{t|t} = R^{-1}_t \), appendix G shows the above updating equations are equivalent to the following constant gain learning algorithm

\[ \theta_t = \theta_{t-1} + g R^{-1}_t x_t (y_t - x'_t \theta_{t-1}) \] \hspace{2cm} (48)
\[ R_t = R_{t-1} + g (x_t x'_t - R_{t-1}) \] \hspace{2cm} (49)

Again to avoid simultaneity between agents’ belief and actual outcomes, we assuming information on data is introduced with a delay in \( \theta_t \). So we actually use

\[ \theta_t = \theta_{t-1} + g R^{-1}_t x_{t-1} (y_{t-1} - x'_{t-1} \theta_{t-1}) \] \hspace{2cm} (50)
\[ R_t = R_{t-1} + g (x_{t-1} x'_{t-1} - R_{t-1}) \] \hspace{2cm} (51)

Agents discount past observations and give relatively more importance to new data, keeping track of the structural changes in the economy. Unlike the learning algorithm with decreasing gain, parameter estimates coming from a constant gain algorithm can not point-converge to a single value even in a time-invariant economy, but they could still converge in distribution around the true value as long as the gain parameter is sufficiently small.

### 7 Quantitative Results

We firstly illustrate the different dynamics of our learning model with the RE model by considering an unexpected one-time positive productivity shock to borrowers’ technology producing tradable output. Secondly, we calibrate our learning model to the U.S. economy and show the model is capable of generating boom and bust in collateral prices following a sequence of unexpected real interest rate reduction. In addition, we show the role of collateral constraint as amplification mechanism is strongly strengthened. The learning model could quantitatively account for recent U.S. boom and bust in house prices, household debt and aggregate consumption dynamics during 2001-2008.

The productivity of the farmer is normalized at one, i.e., \( a = 1 \). The standard deviation of productivity shock is 0.01. The annualized real interest rate is set to 3\%. The elasticity of users’ cost with respect to borrowers’ collateral holding is 2.03, i.e., \( \frac{1}{\eta} = 2.03 \), and gain parameter \( g = 0.05 \). The initial beliefs are set to the RE value and initial data moments are set to the asymptotic moments.
7.1 Response to productivity shock

In this subsection, we set \( \tau \), the proportional transaction cost, to 0.16. This implies the steady state loan-to-value (LTV) ratio is close to 84\%. Figure 5 compares the impulse response functions to 1\% standard deviation positive i.i.d productivity shock based on the extended version of RE model and learning model.

Under RE model. In the impact period, a positive shock expands current net worth of borrowers, which increases their demand for collateral. The collateral is then shifted toward borrowers. In equilibrium, users’ cost rises because lenders hold less collateral and their marginal productivities rise. Lending increase due to rise in both expectation of collateral prices and collateral holding by borrowers. Collateral investment are intertemporal complement to each other, so this generates persistence in collateral holding. Persistence in collateral holding implies persistence in users’ cost in equilibrium and hence in collateral prices. After the shock disappears, higher than steady state users’ cost chokes off further rise in collateral prices and borrowers’ collateral holding, so they revert immediately.

As can be seen from simulated impulse response functions in figure 5, after a positive and temporary 1\% productivity shock, collateral price jumps upward by about 0.08\% and so does collateral holding of farmers for the impact period by about 0.4\%. Rising users’ cost of collateral chokes off the further increase of the demand for collateral. Collateral holding and collateral price converges persistently and monotonically back to their steady state.

Our model with learning. Learning about collateral prices introduces additional interesting propagation due to belief revision and the interaction of shifting expectation about future collateral prices, realized collateral prices and credit limit. In the impact period, the learning model has exactly the same response as the RE model for collateral price and landholding, because we assume agents’ initial belief about unknown parameters is at rational expectation level.

After the impact period, realized higher than expected collateral price induces agents revising upward their beliefs. Their expectation about future collateral price becomes more optimistic than that under rational expectation, as can be seen from the lower left panel of figure 5. Based on more optimistic view on expected liquidation value of collateral, the credit limit is relaxed and larger loans are granted. With larger amount of borrowing, the borrowers will have larger demand on collateral and this will boost collateral price further. The realized collateral prices reinforce the initial optimism. The dynamic interaction of beliefs, credit limit and actual prices generates an expansion period of aggregate lending, investment and production.

Associated with transfer of collateral to borrowers is increase in users’ cost due to rising lenders’ marginal productivities. Borrowers start to reduce collateral holding when capital gain of collateral holding falls short of users’ cost. When collateral price falls short of expectation, agents revise their belief downward. The pessimism belief on future liquidation value of collateral induces a tighter credit constraint. This reduces the net worth and collateral demand of the borrowers. The realization of low collateral
price reinforces the initial pessimism. The interaction of credit limit, belief and its realization leads to a faster decrease in collateral holding and prices than that under RE. Prices and quantities exhibits negative momentum, overshooting and cycling around the steady state.

The impulse response functions in our learning model could be seen from figure 5. Initially, the collateral price, collateral holding and lending jump upward by exactly the same amount as under RE. Afterwards, collateral price are adjusted further up and revert at some point around 0.17%, which is about 2.2 times as much as the peak response under rational expectation. All responses exhibit hump-shape. Prices and quantities overshoot and cycle around the steady state. Finally they converge back to steady state slowly. Consistent with existing literature, the productivity shock in models with collateral constraint has relatively small effects on prices and quantities.

We briefly discuss the role of parameterization of the model. For a larger steady state leverage ratio or larger elasticity of users’ cost of collateral with respect to collateral holding, the amplification of prices and quantities will be greater not only because the initial response is boosted but also they helps generating more stronger momentum effect. In addition, the size of amplification depends negatively on the precision of prior belief, or equivalently, positive on the gain parameter.
7.2 Boom and Bust in house prices, debt and aggregate consumption dynamics

We explore the models’ ability generating boom and bust in house prices\(^{19}\), large lending cycles as well as consumption dynamics observed in the data. We consider a sequence of 1\% unexpected reduction in real interest rate\(^{20}\) for 20 quarterly, which captures in a stylized way the low real rate during year 2001 – 2005. We get model predictions for prices and quantities for 2001-2008. Productivity shock is assumed to be zero during the mean time. Appendix H provides more details on simulation of the model in such scenario.

We set the parameter $\tau$ to 0.32. This implies steady state loan-to-value ratio is close to 68\%, which is roughly the loan-to-value ratio of mortgage product around year 2000 as in Ferrero (2011). The productivity gap between the borrowers and lenders \(\frac{(a+\tau)-C}{(a+\tau)}\) is set to 0.5, and the production share of borrowers \(\frac{(a+\tau)K}{Y}\) is 0.5 at the steady state\(^{21}\). Agents’ initial belief is set to the RE value.

Figure 6 displays response to a sequence of 20 quarters unexpected 1\% reduction in real interest rate under RE and our learning model. Under RE, prices and quantities jump upward following real rate reduction. House prices continue to increase due to persistence of prices and further reduction in real rate. It peaks at about 16\% above the steady state. After the interest rate reduction disappears, house prices start to revert.

In contrast, prices and quantities in our learning model increase initially by the same amount. House prices continue building up at a faster pace than under RE. This is due to the interaction of agents’ optimism, credit limit and price realizations. House prices rise further for about 1 year due to momentum, even if the real rate reductions disappear at the end of 2005. The peak of predicted house prices under learning model is about 40\% at year 2006, which is approximately the accumulated price growth in recent house price boom in the US during the same period and about 2.4 times of the peak response under the RE model.

Large variability of house prices in particular boom and bust in collateral prices largely strengthens the role of collateral constraint in aggregate fluctuation. Lending skyrockets due to both house price booms and rising amount of collateral holding by borrowers. The model predicted debt/GDP ratio rises and peaks at about 100\% at year 2006, while in the data the household net credit market assets/GDP ratio increased by about 90\% during this period. Output response will be amplified through redistribution of collateral from credit unconstrained agents to credit constrained but more productive agents. Given zero net supply of debts and collateralized assets, aggregate consumption will be equal to aggregate output. Aggregate output or aggregate consumption will be equal to aggregate output.

\(^{19}\)Davis and Heathcote (2007) show fluctuation in real estate values are primarily driven by change in land prices.

\(^{20}\)This could be taken as a sequence of exogenous changes in lenders’ discount factor $\beta^C$.

\(^{21}\)These are considered as reasonable value in Cordoba and Ripoll (2004).
Figure 6: Response to a sequence of 20 quarters 1% unexpected negative shock to real interest rate

consumption increases and peaks at about 20% above steady state, while in the data cumulated consumption growth is about 18% during this period.

8 Learning model with Alternative Belief System

In this section we consider our learning model with an alternative belief system that agents believe (log deviation of) collateral prices follow a mean-reversion process. This has the same form as an alternative formulation of rational expectation equilibrium (14). Model dynamics, such as E-stability and transition properties are examined. A comparison with the previous belief system is provided. Finally, impulse responses functions are reported to illustrate the model dynamics with this alternative belief.

8.1 Model with Alternative Belief and $\tau = 0$

We come back to the benchmark model that $\tau = 0$ but now with alternative belief that collateral prices follow mean-reversion process. Eliminating $K_{t-1}^{F}$ in equation (12) by
using (13) at period $t - 1$, we obtain

$$\hat{q}_t = z_1 E_t^P \hat{q}_{t+1} - z_2 E_t^P \hat{q}_t + \hat{q}_{t-1} + z_3 \epsilon_t$$ (52)

where $z_1 = \gamma_1 = \frac{1}{R}(1 + \frac{1}{\eta})$, $z_2 = \gamma_2 + \frac{1}{R} = \frac{1}{\eta} + \frac{1}{R}$ and $z_3 = \gamma_3 = \frac{R-1}{R} \frac{1}{\eta}$. The REE solution for collateral price could be derived as following

$$\hat{q}_t = \bar{\beta}^m + \bar{\beta}^p \hat{q}_{t-1} + \bar{\beta}^s \epsilon_t$$ (53)

where $\bar{\beta}^m = 0$, $\bar{\beta}^p = \frac{\eta}{1+\eta}$, and $\bar{\beta}^s = \frac{1}{\eta}$. This is an alternative formulation of equation (14).

Now we assume agents entertain a subjective model that collateral prices follow an AR(1) process as following

$$\hat{q}_t = \beta^m + \beta^p \hat{q}_{t-1} + \epsilon_t$$ (54)

given $\hat{q}_{t-1}$. The regression residuals $\epsilon_t$ are perceived to be identically and independently normally distributed, i.e., $\epsilon_t \sim i.i.N(0, \sigma_e^2)$. Such belief could be viewed as a discrete version of the Ornstein-Uhlenbeck process (OUP) used in financial economics and practice, which is discussed in more details in Appendix I. This process is governed by three parameters, namely, the long-run mean, speed of reversion, and the innovation shock variance. A higher AR(1) coefficient is equivalent to slower speed of reversion of collateral prices.

Agents have uncertainty about the intercept and persistence $(\beta^m, \beta^p)$ and shock precision $\frac{1}{\sigma_e^2}$. As in section 4.2, we assume agents’ uncertainty about $(\beta^m, \beta^p, \frac{1}{\sigma_e^2})$ is summarized in a normal-gamma prior distribution. Optimal behavior implies that agents update their belief by following RLS algorithm

$$\beta_t = \beta_{t-1} + \frac{1}{t + N} R_{t-1}^{-1} x_{t-1}(\hat{q}_{t-1} - x'_{t-1} \beta_{t-1})$$ (55)

$$R_t = R_{t-1} + \frac{1}{t + N} (x_{t-1} x'_{t-1} - R_{t-1})$$ (56)

where $x_t = (1, \hat{q}_{t-1})'$ are the state variables of the learning algorithm and $\beta_t \equiv (\beta^m, \beta^p)'$ stands for the estimated intercept and persistence of asset price. $N$ measures the precision of initial belief.

In real time agents forecast collateral price as $E_t^P \hat{q}_{t-1} = x'_{t-1} \beta_{t-1}$ and $E_t^P \hat{q}_{t+1} = x'_{t} \beta_{t}$. Substituting conditional expectations into equation (52), we obtain the actual law of motion for collateral prices

$$\hat{q}_t = \frac{z_1 \beta^m_t - z_2 \beta^m_{t-1}}{1 - z_1 \beta^p_t} + \frac{1 - z_2 \beta^p_{t-1}}{1 - z_1 \beta^p_t} \hat{q}_{t-1} + \frac{z_3}{1 - z_1 \beta^p_t} \epsilon_t$$ (57)

The T-map mapping from agents’ PLM to ALM is $T(\beta^m, \beta^p) \equiv (T_1, T_2)(\beta^m, \beta^p) \equiv (\frac{(z_1 - z_2) \beta^m}{1 - z_1 \beta^p}, \frac{1 - z_2 \beta^p}{1 - z_1 \beta^p})$. Local stability of the MSV REE is determined by the stability of
the following associated ODEs

\[
\begin{align*}
\frac{d\beta^m}{d\tau} & = \frac{(z_1 - z_2)\beta^m}{1 - z_1\beta^p} - \beta^m \\
\frac{d\beta^p}{d\tau} & = \frac{1 - z_2\beta^p}{1 - z_1\beta^p} - \beta^p
\end{align*}
\]

The following condition establishes a sufficient condition for E-stability of the MSV equilibrium (53).

**Proposition 10**

The MSV equilibrium (53) for the economy represented by equation (52) is E-stable for all admissible parameters in $\Theta_0$.

**Proof.** see Appendix J. ■

Again the users’ cost of collateral plays important role in stabilizing collateral demand and prices around the neighborhood of the REEequilibrium. Reformulat equation (31) as

\[
\hat{K}_t^F = \frac{R}{R - 1} \left[\frac{1}{R} E_t^P \hat{q}_{t+1} - E_t^P \hat{q}_t \right] + \hat{K}_{t-1}^F
\]  
(58)

The E-stability condition could be understood as following. Consider a small deviation of collateral price expectation above and away from Rational expectation level and fix agents’ belief about persistence of collateral prices at $(\beta^m, \beta^p)$, which are above their RE value. Conditional expectations are $E_t^P \hat{q}_t = \beta^m + \beta^p \hat{q}_{t-1}$ and $E_t^P \hat{q}_{t+1} = \beta^m + \beta^p \hat{q}_t = \beta^m + \beta^p T_1(\beta^m, \beta^p) + \beta^p T_2(\beta^p) \hat{q}_{t-1}$. Note $T_1(\beta^m, \beta^p) = \frac{(z_1 - z_2)\beta^m}{1 - z_1\beta^p} < 0$ because $z_1 < z_2$.

As long as $T_2(\beta^p)$ is smaller than $R$, which is the case around the neighborhood of RE equilibrium, we have $\frac{1}{R} E_t^P \hat{q}_{t+1} < E_t^P \hat{q}_t$ for any admissible parameters. The users’ cost of collateral will eventually outweigh the capital gain. Roughly speaking, given that the E-stability condition is satisfied and the estimates are around the neighborhood of steady state, we have $\beta_t \to \bar{\beta}$ and $\nu_t \to \infty$ almost surely.

We discuss the deterministic dynamics of the learning model by assuming $\epsilon_t = 0$ for all $t$. A simplified PLM without learning about the steady state $\bar{\beta}^m$, that is, $\hat{q}_t = \beta^p \hat{q}_{t-1} + t_t$ is considered. Appendix K shows the T-map mapping agents’ perceived persistence to the actual persistence coefficient could be expressed by

\[
T_2(\beta^p_{t-1}, \beta^p_t) = \frac{1 - z_2\beta^p_{t-1}}{1 - \beta^p_{t-1} z_1} = \beta^p + \frac{z_2(\beta^p - \beta^p_{t-1}) - \beta^p z_1(\beta^p - \beta^p_t)}{1 - \beta^p z_1}
\]
where $\beta^p_t$ is the RE value the persistence coefficient. It can be seen that the persistence in the actual law of motion for collateral prices will depend on both $\beta^p_{t-1}$ and $\beta^p_t$. The following proposition examines under which parameterizations our learning model with alternative belief displays momentum in beliefs.

**Proposition 11**

For all admissible parameters in $\Theta_0$, learning model with alternative beliefs displays momentum in beliefs, i.e.,

1. if $\beta^p_t \leq \beta^p$ and $\beta^p_t > \beta^p_{t-1}$, then $\beta^p_{t+1} > \beta^p_t$.

2. if $\beta^p_t \geq \beta^p$ and $\beta^p_t < \beta^p_{t-1}$, then $\beta^p_{t+1} < \beta^p_t$.

**Proof.** see Appendix K. ■

When the agents’ belief about persistence of collateral prices arrived at the RE level from below (above), i.e., $\beta^p_t > \beta^p_{t-1}$, $\beta^p_t < \beta^p_{t-1}$, the realized persistence implied by the above T-map would exceed (below) the RE level. Using realized collateral prices, the agents update their belief about persistence upward (downward), i.e., $\beta^p_{t+1} > \beta^p_t$, $(\beta^p_{t+1} < \beta^p_t)$.

### 8.2 Comparison of Two Belief Systems

We compare differences of the learning models with the first belief system (18) and alternative belief (54). We show momentum generated by learning model with the first belief system is less pronounced. Recall the T-map mapping perceived persistence to actual persistence with alternative belief is $T_2(\beta^p_{t-1}, \beta^p_t) = \frac{1-\gamma_2 \beta^p_{t-1}}{1-\gamma_1 \beta^p_t}$. The elasticity of the persistence coefficient in objective collateral prices process with respect to agents’ current belief about collateral price persistence at the steady state is

$$\frac{\partial T_2(\beta^p_{t-1}, \beta^p_t)}{\partial \beta^p_t} \bigg|_{\beta^p_{t-1}=\beta^p_t=\beta^p} = \frac{1}{R-1}$$

With the first belief, recall the T-map $T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2 \phi^p_{t-1}}{1-\gamma_1 \phi^p_t}$. The elasticity of actual parameters with respect to agents’ current belief

$$\frac{\partial T_2(\phi^p_{t-1}, \phi^p_t)}{\partial \phi^p_t} \bigg|_{\phi^p_{t-1}=\phi^p_t=\phi^p} = \frac{R(1+\eta) - \eta}{R(1+\eta)} \frac{1}{R-1}$$

The elasticity measures the sensitivity of actual parameter to small change of parameter belief away from rational expectation level. Easy to see the latter elasticity (60) is smaller than the former (59) for any admissible parameterizations and increasing in $\frac{1}{\eta}$.
Figure 7: Response to one standard deviation i.i.d positive productivity shock: alternative belief

### 8.3 Extended Model \((\tau \geq 0)\) and Modification of Beliefs

Now we are back to the model with \(\tau \geq 0\). Combining equations (40) and (13) leads to

\[
\hat{q}_t = \delta_1 E^P_t \hat{q}_{t+1} - \delta_2 E^P_{t-1} \hat{q}_t + \hat{q}_{t-1} + \delta_3 \epsilon_t
\]

where \(\delta_1 = \xi_1\), \(\delta_2 = \xi_2 + \frac{1}{R} = (1 - \tau) \frac{1}{\eta} \frac{R - 1}{R} - \frac{1}{R(1 - \tau)} + \frac{1}{R}\) and \(\delta_3 = \xi_4\).

The MSV REE is

\[
\hat{q}_t = \bar{\alpha}^m + \bar{\alpha}^p \hat{q}_{t-1} + \bar{\alpha}^s \epsilon_t
\]

where \(\bar{\alpha}^m = 0\), \(\bar{\alpha}^p = \frac{1 + \frac{1}{R} - \frac{1}{R(1 - \tau)}}{1 + \frac{1}{R} - \frac{1}{R(1 - \tau)}}\) and \(\bar{\alpha}^s = \frac{1}{\eta} \frac{R - 1}{R(1 - \tau)}\).

In the learning model agents’ conditional expectations are \(E^P_t \hat{q}_{t+1} = \alpha^m_t + \alpha^p_t \hat{q}_t\) and \(E^P_{t-1} \hat{q}_t = \alpha^m_{t-1} + \alpha^p_{t-1} \hat{q}_{t-1}\). The actual law of motion for collateral prices are

\[
\hat{q}_t = \frac{\delta_1 \alpha^m_t - \delta_2 \alpha^m_{t-1}}{1 - \delta_1 \alpha^p_t} + \frac{1 - \delta_2 \alpha^p_{t-1}}{1 - \delta_1 \alpha^p_t} \hat{q}_t + \frac{\delta_3}{1 - \delta_1 \alpha^p_t} \epsilon_t
\]

Part (1) of the following proposition examines the E-stability condition for The REE equilibrium (62). We investigate the deterministic dynamics of learning model when \(\epsilon_t = 0\) and agents perceive \(E^P_t \hat{q}_{t+1} = \alpha^p_t \hat{q}_t\) after adding proportional transaction cost \(\tau\).
Proposition 12

For any admissible parameters in $\Theta_1$,

1. The REE equilibrium (62) is E-stable.

2. Momentum in beliefs always hold holds.

**Proof.** The proof of the first part is in appendix L and the second part could be proved following the proof of proposition 2 by noting $\delta_2 > \delta_1$. ■

In addition, as in section 6.2, we assume agents perceive the fundamental parameters are drifting over time, i.e., random walk. Optimal behavior implies agents’ beliefs are updated by following constant-gain learning algorithm

$$
\begin{align*}
\alpha_t &= \alpha_{t-1} + g R_t^{-1} x_{t-1} ('tilt_{t-1} - x^\prime_{t-1} \alpha_{t-1}) \\
R_t &= R_{t-1} + g (x_{t-1} x_{t-1}^\prime - R_{t-1})
\end{align*}
$$

where $\alpha_t = (\alpha_t^m, \alpha_t^p)$, and $x_t = (1, \tilde{q}_{t-1})$.

8.4 IRFs to productivity shock with alternative belief system

The parameters are the same as under the first belief system and $\tau = 0.16$, except the gain parameter is set to a much smaller value, i.e., 0.0039, due to stronger momentum effect under this belief system. Figure 7 displays the impulse response functions to a positive and one-time productivity shock. The interaction of agents’ belief, credit limit and asset prices generate similar dynamics in prices and quantities as the previous belief system that agents use collateral holding to forecast future collateral prices. The magnitude of impulse response functions is similar to these generated in figure 5.

9 Conclusion

We develop an model with collateral constraint in which agents possess imperfect knowledge and learn about collateral prices process. It could quantitatively account for recent US boom and bust in house price, debt and aggregate consumption dynamics during 2001-2008. The model suggests sizable asset prices booms may arise from low level of interest rate, which confirms the point made in Adam, Kuang and Marcet (2011) but through different model specification and belief systems.

We also find highly leveraged economy may be more prone to self-reinforcing learning dynamics when agents’ subjective beliefs are allowed to have small deviation from REE beliefs. It may help to understand economic volatilities of aggregate variables
across regimes with different leverages or cross-countries. For example, in studying the behavior of money, credit and macroeconomic indicators for 14 countries over the year 1870-2008, Schularick and Taylor (2011) find output losses today is as large as Pre-WW2 despite more activist policies and the presence of deposit insurance and allude the important role of increased leverage of financial sector. Secondly, our model provides additional rationale for reasonable capital requirement regulation.

Asset prices/value play large role in aggregate fluctuation through many channels such as household, corporate balance sheet, bank capital channel, etc. It is interesting to study further and quantify the role of the interaction of agents’ uncertainty in financial market in particular imperfect knowledge with other kinds of credit frictions in aggregate fluctuations. Also interested is to look into how the uncertainty in the financial markets interacts with economic agents’ decisions in other markets, such as labor market. Finally, our model facilitates the discussion of how monetary policies could affect whether a bubble builds up in the first place and can affect the speed at which it deflates, as well as approriate design of policies to stabilizing the economy and financial system.


A Log-linearization of the Benchmark Model

The following system of equations represents the dynamics of the economy

\[ K_t^F(i) = \frac{(a + \epsilon_t a + q_t - E_t^P q_t)}{q_t - \frac{1}{\eta_t} E_t^P q_{t+1}} K_{t-1}^F(i) \]  \hspace{2cm} (66)

\[ u_t^i = q_t - \frac{1}{R} E_t^P q_{t+1} = \frac{G'(K_t^G)}{R} \]  \hspace{2cm} (67)

The steady state value of the endogenous variables are

\[ q^* = a R, u^* = a, K^G = G_0^* \]  \hspace{2cm} (68)

\[ K^F = \tilde{K} - K^G, b^F = qK^F/R \text{ and } c^F = \tilde{c}K^F. \]

Log-linearization of equation (67) leads to

\[ \tilde{u}_t = \frac{1}{\eta_t} \tilde{K}_t^F = \frac{R}{R-1} (\tilde{q}_t - \frac{1}{R} E_t^P \tilde{q}_{t+1}) \]  \hspace{2cm} (69)

Log-linearization of equation (66) leads to

\[ \tilde{K}_t^F = \frac{R}{R-1} [(\tilde{q}_t - E_t^P \tilde{q}_t) - (\tilde{q}_t - \frac{1}{R} E_t^P \tilde{q}_{t+1})] + \tilde{K}_{t-1}^F + \epsilon_t \]  \hspace{2cm} (70)

Plugging equation (68) into (69), we arrive at

\[ \tilde{q}_t = \gamma_1 E_t^P \tilde{q}_{t+1} - \gamma_2 E_{t-1}^P \tilde{q}_t + \gamma_3 (\tilde{K}_{t-1}^F + \epsilon_t) \]

where \( \gamma_1 = \frac{1}{R}(1 + \frac{1}{\eta_t}), \gamma_2 = \frac{1}{\eta_t}, \gamma_3 = \frac{R-1}{R} \frac{1}{\eta_t}. \)

B Derivation of the Bayesian Posterior Mean

We assume the prior distribution of parameters \((\theta^m, \theta^p, H)\) is a Normal-Gamma distribution as following

\[ H \sim G(\gamma_0, s_0^{-2}) \]

\[ (\theta^m, \theta^p)' \mid H \sim N((\theta_0^m, \theta_0^p)', (\nu_0 H)^{-1}) \]

The prior distribution for the precision of the shock is gamma distribution and the conditional prior of \((\theta^m, \theta^p)\) given the precision of the shock is multivariate normal distribution.

We drop the terms which do not involve \((\theta, h)\) by using the proportionality symbol. The conditional probability of the asset price is a normal distribution with following conditional probability density function

\[ p(y_t | \theta, h) \propto h^{\frac{1}{2}} \exp\left\{-\frac{h}{2}(y_t - x_t' \theta)'(y_t - x_t' \theta)\right\} \]

The prior density of the parameters is following
\[ p(\theta, h) \propto h^{\gamma t-1} \exp\{-s_{t-1}^{-2}h\} \nu_{t-1}^{1/2} \exp\{-\frac{h}{2}(\theta - \theta_{t-1})'\nu_{t-1}(\theta - \theta_{t-1})\} \]

We show the posterior distribution of the parameters are following:

\[
\begin{align*}
\theta | H & \sim N(\nu_t, (\nu_t h)^{-1}) \\
H & \sim G(\gamma_t, s_t^{-2})
\end{align*}
\]

with probability density function

\[ p(\theta, h|y_t) \propto h^{\gamma t-1} \exp\{-s_{t-1}^{-2}h\} \nu_{t-1}^{1/2} \exp\{-\frac{h}{2}(\theta - \theta_t)'\nu_t(\theta - \theta_t)\} \]

where

\[
\begin{align*}
\theta_t &= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1}) \\
\nu_t &= \nu_{t-1} + x_t x_t' \\
\gamma_t &= \gamma_{t-1} + \frac{1}{2} \\
\nu_{t-1} &= \frac{1}{2} (y_t - x_t' \theta_{t-1})' (x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} (y_t - x_t' \theta_{t-1})
\end{align*}
\]

The above equations could be derived using Bayes’ law. The critical intermediate steps are presented here. We have

\[ p(\theta, h|y_t) \propto p(y_t|\theta, h)p(\theta, h) \]

The posterior mean of the parameters is

\[
\begin{align*}
\theta_t &= (x_t x_t' + \nu_{t-1})^{-1} (\nu_{t-1} \theta_{t-1} + x_t y_t) \\
&= (x_t x_t' + \nu_{t-1})^{-1} \nu_{t-1} \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
&= \theta_{t-1} - (x_t x_t' + \nu_{t-1})^{-1} x_t \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t y_t \\
&= \theta_{t-1} + (x_t x_t' + \nu_{t-1})^{-1} x_t (y_t - x_t' \theta_{t-1})
\end{align*}
\]

Note

\[
\begin{align*}
(y_t - x_t' \theta)' (y_t - x_t' \theta) + (\theta - \theta_{t-1})' \nu_{t-1} (\theta - \theta_{t-1}) \\
&= y_t y_t - 2\theta' x_t y_t + \theta' x_t x_t' \theta + \theta' \nu_{t-1} \theta - 2\theta' \nu_{t-1} \theta_{t-1} + \theta_{t-1}' \nu_{t-1} \theta_{t-1} \\
&= \theta' (x_t x_t' + \nu_{t-1}) \theta - 2\theta' (x_t y_t + \nu_{t-1} \theta_{t-1}) + y_t y_t + \theta_{t-1}' \nu_{t-1} \theta_{t-1} \\
&= (\theta - (x_t x_t' + \nu_{t-1})^{-1} (\nu_{t-1} \theta_{t-1} + x_t y_t))' (x_t x_t' + \nu_{t-1}) (\theta - (x_t x_t' + \nu_{t-1})^{-1} (\nu_{t-1} \theta_{t-1} + x_t y_t)) \\
&\quad [y_t y_t + \theta_{t-1}' \nu_{t-1} \theta_{t-1} - (\nu_{t-1} \theta_{t-1} + x_t y_t)' (x_t x_t' + \nu_{t-1})^{-1} (\nu_{t-1} \theta_{t-1} + x_t y_t)]
\end{align*}
\]
C Proof of proposition 4

Recall local stability of the MSV RE is determined by the stability of the following associated ODEs

\[
\frac{d\phi^m}{d\tau} = T_1(\phi^m, \phi^p) - \phi^m
\]
\[
\frac{d\phi^p}{d\tau} = T_2(\phi^p, \phi^p) - \phi^p
\]

where

\[
T_1(\phi^m_{t-1}, \phi^m_t, \phi^p_t) = \frac{(\gamma_1 \phi^m_{t-1} - \gamma_2 \phi^m_t)(1 + \frac{\phi^p_{t-1}}{\gamma_3 R})}{1 + \frac{\phi^p_{t-1}}{\gamma_3 R} - \gamma_1 \phi^p_t \frac{1}{\gamma_3}}, \quad T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2 \phi^p_{t-1}}{1 - \gamma_1 \phi^p_t \frac{1}{\gamma_3}}
\]

\[\gamma_1 = \frac{1}{R}(1 + \frac{1}{\eta}), \quad \gamma_2 = \frac{1}{\eta}, \quad \text{and} \quad \gamma_3 = \frac{R-1}{R} \frac{1}{\eta}.\]

The rational expectation solution is

\[\hat{q}_t = \bar{\phi}^m + \bar{\phi}^p K^F_{t-1} + \bar{s} \epsilon_t \quad (71)\]

where \(\bar{\phi}^m = 0, \bar{\phi}^p = \frac{R-1}{R} \frac{1}{\eta+1-\frac{1}{R}}\) and \(\bar{s} = \frac{1}{\eta}\).

The E-stability for the above MSV equilibrium requires the eigenvalues of the Jacobian of the right hand side is negative. Since \(\phi^m\) does not show up in the ODE for \(\phi^p\), the eigenvalues will be on the diagonal and only two partial derivatives, i.e., \(\frac{\partial T_1(\phi^m, \phi^p)}{\partial \phi^m}|_{\phi^m=\bar{\phi}^m, \phi^p=\bar{\phi}^p}\) and \(\frac{\partial T_2(\phi^p, \phi^p)}{\partial \phi^p}|_{\phi^p=\bar{\phi}^p}\), matter for the E-stability.

Plugging the parameters \(\gamma's\), we obtain

\[T_2(\phi^p, \phi^p) = \frac{\frac{1}{\eta}(\phi^p - \frac{R-1}{R})}{\frac{\phi^p - (\frac{R-1}{R})}{\eta \phi^p + (\frac{R-1}{R})}} = \frac{(\phi^p - \frac{R-1}{R})}{\eta \phi^p + (\frac{R-1}{R})}
\]

The second derivative

\[\frac{\partial T_2(\phi^p, \phi^p)}{\partial \phi^p}|_{\phi^p=\bar{\phi}^p} = \frac{\phi^p - \frac{R-1}{R} + \frac{R-1}{\eta}}{\phi^p - (\frac{R-1}{R})} = \frac{\eta R - \eta + R(1 + \frac{1}{\eta} - \frac{1}{R})}{1 - R(\eta + 1 - \frac{\eta}{R})}
\]

The denominator is negative and we show the numerator is positive. The numerator is positive is equivalent to

\[\frac{\eta R}{\eta} - \eta + R + \frac{R}{\eta} - 1 > 0\]
\[(\eta - R)^2 + R^2 \eta > 0\]
which holds for all admissible parameters. Note
\[
T_1(\phi^m, \phi^p) = \frac{(\gamma_1 - \gamma_2 (1 + \frac{\phi^p}{\gamma_3 R})) \phi^m}{1 + \frac{\phi^p}{\gamma_3 R} - \gamma_1 \phi^p \frac{1}{\gamma_3}}
\]

Given \(T_1(\phi^m, \phi^p) = \phi^m = 0\), the first derivative
\[
\frac{\partial T_1(\phi^m, \phi^p)}{\partial \phi^m} |_{\beta^m = \beta^m, \beta^p = \beta^p}\]
\[
= \frac{(\gamma_1 - \gamma_2 (1 + \frac{\phi^p}{\gamma_3 R}))}{1 + \frac{\phi^p}{\gamma_3 R} - \gamma_1 \phi^p \frac{1}{\gamma_3}}
\]
\[
= \frac{(\gamma_1 - \gamma_2) \gamma_3 R - \gamma_2 \phi^p}{\gamma_3 R + \phi^p (1 - R \gamma_1)}
\]
\[
= \frac{(1 + \frac{1}{\eta} - \frac{R}{\eta})(\eta + 1 - \frac{R}{\eta}) - 1}{(\eta + 1 - \frac{R}{\eta}) R - 1}
\]

Since \(1 + \frac{1}{\eta} - \frac{R}{\eta} < R\) holds for all admissible parameters, we have \(\frac{\partial T_1(\phi^m, \phi^p)}{\partial \phi^m} |_{\beta^m = \beta^m, \beta^p = \beta^p} < 1\). To sum up, the MSV equilibrium is E-stable for all admissible parameters.

D Proof of proposition 6

Recall the T-map on agents’ perceived slope coefficients \(T_2(\phi^p_{t-1}, \phi^p_t)\).
\[
T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\gamma_3 - \gamma_2 \phi^p_{t-1}}{1 - \gamma_1 \phi^p_{t-1} \frac{1}{\gamma_3}}
\]

(72)

The above expression could be transformed to the following by substituting \(\gamma_1, \gamma_2\) and \(\gamma_3\) and simplifying
\[
T_2 = \frac{R-1}{R} - \phi^p_{t-1}
\]

(73)

where \(f(\phi^p_t) \equiv 2 - \frac{(\eta+1)(R-1)}{\eta \phi^p_t + R-1}\). Further algebra yields
\[
T_2(\phi^p_{t-1}, \phi^p_t) = \frac{\phi^p - \phi^p (1 - f(\phi^p_t)) + (\frac{R-1}{R} - \phi^p_{t-1})}{1 - f(\phi^p_t)}
\]

(74)

\[
= \frac{\phi^p - \phi^p (1 - f(\phi^p_t)) + \phi^p (1 - f(\phi^p_t)) - (\frac{R-1}{R} - \phi^p) + (\frac{R-1}{R} - \phi^p_{t-1})}{1 - f(\phi^p_t)}
\]

(75)

\[
= \frac{(\phi^p - \phi^p_{t-1}) - \phi^p (f(\phi^p_t) - f(\phi^p_{t-1}))}{1 - f(\phi^p_t)}
\]

(76)

\[
\simeq \frac{(\phi^p - \phi^p_{t-1}) - \phi^p f'(\phi^p)(\phi^p - \phi^p_{t-1})}{1 - f(\phi^p_t)}
\]

(77)
In equation (75), we use \( \bar{\phi}^p \) is the fixed point of the \( T_2 \) map, i.e., \( \bar{\phi}^p = \frac{R^{-1} - \bar{\phi}^p}{1 - f(\bar{\phi}^p)} \). In last step we do the first-order Taylor approximation of function \( f \) at \( \bar{\phi} \). A sufficient condition to guarantee momentum is \( 1 > \bar{\phi}^p f'(\bar{\phi}^p) \). It could be shown given \( \phi_{t-1}^p < \phi_t^p < \bar{\phi} \), we have \( T_2(\phi_{t-1}^p, \phi_t^p) > \bar{\phi} \). Using the belief updating equation, we have \( \phi_{t+1}^p > \phi_t^p \). Similarly, given \( \phi_{t-1}^p > \phi_t^p > \bar{\phi} \), we have \( T_2(\phi_{t-1}^p, \phi_t^p) < \bar{\phi} \). Then we have \( \phi_{t+1}^p < \phi_t^p \).

It could be shown that the condition \( 1 > \bar{\phi}^p f'(\bar{\phi}^p) \) is equivalent to

\[
(R - \frac{\eta}{2})^2 > \eta^2 \left( \frac{1}{4} - \frac{1}{1 + \eta} \right)
\]

Case 1: if \( \frac{1}{4} < \frac{1}{1 + \eta} \) or \( \frac{1}{\eta} > \frac{1}{3} \), (78) is satisfied. Case 2: if \( \frac{1}{4} > \frac{1}{1 + \eta} \) or \( \eta > 3 \), then we have either \( R < \frac{\eta}{2} \left( 1 - \sqrt{1 - \frac{4}{1 + \eta}} \right) \) or \( R > \frac{\eta}{2} \left( 1 + \sqrt{1 - \frac{4}{1 + \eta}} \right) \). The latter is dropped because it will imply the gross real rate \( R > 1.5 \).

## E Proof of Proposition 8

Local stability of the MSV RE is determined by the stability of the following associated ODEs

\[
\frac{d\zeta^m}{d\tau} = T_1(\zeta^m, \zeta^p) - \zeta^m
\]
\[
\frac{d\zeta^p}{d\tau} = T_2(\zeta^p, \zeta^p) - \zeta^p
\]

where

\[
T_1(\zeta^m, \zeta^p) = \frac{(\xi_1 - \xi_3 \zeta^m)(1 + \frac{\zeta^p}{\zeta^p_R}) - \xi_3 \zeta^m \zeta^p}{1 + \frac{\zeta^p}{\zeta^p_R} - \xi_1 \zeta^p},
\]
\[
T_2(\zeta^p, \zeta^p) = \frac{\xi_3 - \xi_2 \zeta^p}{1 - \frac{\zeta^p}{\zeta^p_R} \xi_3},
\]

\( \xi_2 = (1 - \tau) \frac{1 - R^{-1}}{\eta R^{-1}} \), \( \xi_3 = \frac{1}{\eta R^{-1}} \), \( \zeta^m = 0 \), and \( \zeta^p = \frac{1}{\eta R^{-1}} \).

The E-stability condition requires the eigenvalues of the Jacobian of the right hand side is negative. Since \( \zeta^m \) does not show up in the ODE for \( \zeta^p \), the eigenvalues will be on the diagonal and only two partial derivatives, i.e., \( \frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m} \big|_{\zeta^m=\zeta^m, \zeta^p=\zeta^p} \) and \( \frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p} \big|_{\zeta^p=\zeta^p} \), matter for the E-stability.

\[
T_2(\zeta^p, \zeta^p) = \frac{\xi_3 - \xi_2 \zeta^p}{1 - \frac{\zeta^p}{\zeta^p_R} \xi_3}
\]
\[
= \frac{(\xi_3 + \frac{\zeta^p}{\zeta^p_R})(\xi_3 - \xi_2 \zeta^p)}{\xi_3 + (\frac{1}{\zeta^p_R} - \xi_1) \zeta^p}
\]

\[22\]The gross interest rate \( R \) should not be too large, otherwise the assumption 2 of KM model will be violated. Assumption 2 guarantees that borrowers will not want to consume more than the bruised consumption good.
The derivative of $T_2$ with respect to $\zeta^p$ is

$$\left. \frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p}\right|_{\zeta^p = \zeta^p} = \frac{(\frac{1}{R} - \xi_2)\xi_3 - \zeta^p(\frac{2}{R}\xi_2 + \frac{1}{R} - \xi_1)}{\xi_3 + (\frac{1}{R} - \xi_1)\zeta^p}$$

We proceed to show the above expression is smaller than 1.

$$\left. \frac{\partial T_2(\zeta^p, \zeta^p)}{\partial \zeta^p}\right|_{\zeta^p = \zeta^p} < 1$$

is equivalent to

$$(\frac{1}{R} - \xi_2)\xi_3 - \zeta^p(\frac{2}{R}\xi_2 + \frac{1}{R} - \xi_1) < \xi_3 + (\frac{1}{R} - \xi_1)\zeta^p$$

Rearranging the above inequality yields

$$(\frac{1}{R} - \xi_2 - 1)\xi_3 < 2\zeta^p(\frac{\xi_2}{R} + \frac{1}{R} - \xi_1)$$

Plugging parameters, we could show the left hand side is negative and right hand side is zero.

Now we look at the first derivative. $T_1(\zeta^m, \zeta^p) = \frac{\xi_1\zeta^m - \xi_2\zeta^m(1 + \frac{\zeta^p}{\xi_3})}{1 + \frac{\zeta^p}{\xi_3} - \xi_1\zeta^m \frac{1}{\xi_3}}$ and

$$\left. \frac{\partial T_1(\zeta^m, \zeta^p)}{\partial \zeta^m}\right|_{\zeta^m = \zeta^m, \zeta^p = \zeta^p} = \frac{\xi_1 - \xi_2(1 + \frac{\zeta^p}{\xi_3})}{1 + \frac{\zeta^p}{\xi_3} - \xi_1\zeta^m \frac{1}{\xi_3}} = \frac{(\xi_1 - \xi_2)\xi_3 R - \xi_2\zeta^p}{\xi_3 R + \zeta^p(1 - \xi_1 R)}$$

Note the denominator of the above derivative is positive. Now we show the above derivative is smaller than 1, which is equivalent to show

$$(\xi_1 - \xi_2)\xi_3 R - \xi_2\zeta^p < \xi_3 R + \zeta^p(1 - \xi_1 R)$$

Rearranging the above inequality yields

$$(\xi_1 - \xi_2 - 1)\xi_3 R < (\xi_2 + 1 - \xi_1 R)\zeta^p$$

Plugging the parameters, we could show the left hand side of the above inequality is negative and the right hand side is zero.
F  Proof of Proposition 9

Define \( s(\zeta_t^p) = \frac{\xi_1 \zeta_t^p}{\xi_3 + \xi_4^t} \). Recall the T-map is

\[
T_2(\zeta_{t-1}^p, \zeta_t^p) = \frac{\xi_3 - \xi_2 \zeta_{t-1}^p}{1 - s(\zeta_t^p)}
\]

\[
= \frac{\xi_3 - \xi_2 \zeta_{t-1}^p}{1 - s(\zeta_t^p)}
\]

\[
\approx \zeta_t^p + \frac{\xi_2 (\zeta_t^p - \zeta_{t-1}^p) - s(\zeta_t^p) s(\zeta_t^p) - s(\zeta_t^p)}{1 - s(\zeta_t^p)}
\]

Following proof of proposition 11, a sufficient condition to guarantee momentum in belief is \( \xi_2 > \zeta_t^p s'(\zeta_t^p) \). Plugging the parameters, this condition is equivalent to

\[
(R^2(R - 1) - \eta R(R - 1) - \eta^2(1 - \tau)^2 + R\eta(R(R - 1) + 2\eta)(1 - \tau) - R^2\eta^2 > 0
\]

It could be simplified to

\[
(\eta(R - (1 - \tau)) - \frac{R}{2}(R - 1)(1 - \tau)^2) < R^2(1 - \tau)^2((R - 1) + \frac{(R - 1)^2}{4})
\]

Case 1: assume \( \eta(R - (1 - \tau)) > \frac{R(R-1)(1-\tau)}{2} \), we get \( \frac{2(R-1)(1-\tau)}{R(R-1)(1-\tau)} > \frac{R}{\eta(R)} > \frac{R}{g(R)} \). Case 2: assume \( \eta(R - (1 - \tau)) < \frac{R(R-1)(1-\tau)}{2} \), we obtain \( \frac{1}{\eta} > \max\{\frac{2(R-1)(1-\tau)}{R(R-1)(1-\tau)}, \frac{R}{g(R)}\} \).

Given \( \frac{2(R-1)(1-\tau)}{R(R-1)(1-\tau)} > \frac{R}{g(R)} \) for all admissible parameters, we arrive at \( \frac{1}{\eta} > \frac{R}{g(R)} \), which is equivalent to \( \frac{1}{R^2} > \frac{1}{g(R)^2 + 1} \); where \( g(R) = R(\sqrt{R - 1} + \frac{(R-1)^2}{4} + \frac{R-1}{2}) \). It could be shown that when \( \tau = 0 \), our condition here will collapse to the condition in proposition 9.

G  Deriving the constant-gain learning algorithm from Bayesian updating

Agents perceive the following random walk model of coefficient variation

\[
\tilde{\theta}_t = \tilde{\theta}_{t-1} + \epsilon_t \\
\tilde{q}_t = \tilde{\theta}_{t-1} x_t + \zeta_t \\
E_t \epsilon_t' = R_{1t} \tag{79}
\]

\[
E_t q_t' = R_{2t} \tag{80}
\]
Define $P_{t-1} = E[(\hat{\theta}_{t-1} - \theta_{t-1})(\hat{\theta}_{t-1} - \theta_{t-1})]$. The prior belief about $\hat{\theta}_0$ are $N(\theta_0, P_{0/0})$.

The posterior of $\theta_t$ could be represented by the basic Kalman filter, which takes the form of following recursions$^{23}$

\[
\hat{\theta}_t = \hat{\theta}_{t-1} + L_t[y_t - x_t^T \hat{\theta}_{t-1}] \\
L_t = \frac{P_{t|t-1} x_t}{R_{2t} + x_t^T P_{t|t-1} x_t} \\
P_{t+1|t} = P_{t|t-1} - \frac{P_{t|t-1} x_t x_t^T P_{t|t-1}}{R_{2t} + x_t^T P_{t|t-1} x_t} + R_{1t+1}
\]

We assume agents perceive $R_{1t} = \frac{g}{1-g} P_{t-1|t-1}$ and $R_{2t} = \frac{1}{g}$. Note $P_{t|t-1} = P_{t-1|t-1} + R_{1t} = \frac{1}{1-g} P_{t-1|t-1}$.

Equations (82) – (83) become

\[
\hat{\theta}_t = \hat{\theta}_{t-1} + L_t[y_t - x_t^T \hat{\theta}_{t-1}] \\
L_t = \frac{P_{t-1|t-1} x_t}{g + x_t^T P_{t-1|t-1} x_t}
\]

And equation (81) becomes

\[
P_{t+1|t} - R_{1t+1} = P_{t|t-1} - \frac{P_{t|t-1} x_t x_t^T P_{t|t-1}}{R_{2t} + x_t^T P_{t|t-1} x_t} \\
\]

The constant gain learning algorithm is following

\[
\hat{\theta}_t = \hat{\theta}_{t-1} + g R^{-1}_{t-1} x_t (y_t - \hat{\theta}_{t-1}^T x_t) \\
R_t = R_{t-1} + g (x_t x_t^T - R_{t-1})
\]

Below we show the above two formulations are equivalent. Use $R_t^{-1} \equiv P_{t|t}$, equation (89) yields

\[
R_t^{-1} = ((1-g)R_{t-1} + gx_t x_t^T)^{-1} \\
= \frac{1}{1-g} P_{t-1}^{-1} - \frac{1}{1-g} R_{t-1}^{-1} x_t \left[ x_t^T \frac{1}{1-g} R_{t-1}^{-1} x_t + \frac{1}{g} \right]^{-1} x_t^T \frac{1}{1-g} R_{t-1}^{-1} \\
= \frac{1}{1-g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1} x_t x_t^T P_{t-1|t-1}}{x_t^T P_{t-1|t-1} x_t + \frac{1}{g}} \right]
\]

$^{23}$Note for the model considered here we have $\theta_{t-1} = \theta_{t-1|t-1}$, so we suppress the conditioned information set and use $\theta_t$ for both.
From equation (90) to equation (91), we use the matrix inversion formula, which is stated in lemma 1 below. Specifically, we apply lemma 1 with \( A = (1 - g)R_{t-1} \), \( B = x_t \), \( C = g \), \( D = x_t' \).

Now we proceed to match equation (84) and (88). It suffices to show that \( gR_t^{-1}x_t = L_t \).

\[
gR_t^{-1}x_t = gP_{t|t}x_t = \frac{g}{1-g} \left[ P_{t-1|t-1} - \frac{P_{t-1|t-1}x_t'P_{t-1|t-1}x_t}{1-g} + x_t'P_{t-1|t-1}x_t \right] = L_t \tag{93}
\]

From equation (93) to (94), we use equation (87).

**Lemma 1.** Let \( A, B, C \) and \( D \) be matrices of compatible dimensions, so that the product \( BCD \) and the sum \( A + BCD \) exist. Then

\[
[A + BCD]^{-1} = A^{-1} - A^{-1}B[DA^{-1}B + C^{-1}]^{-1}DA^{-1} \tag{97}
\]

Proof: see Ljung and Soederstroem (1983) pp. 19. (Sketch: show the RHS of (97) multiplied by \( A + BCD \) from the right is equal to identity matrix.)

**H Learning Model with Unexpected Real Interest Reduction**

This appendix provides details on the learning model based on which the impulse response function for a sequence of real rate reduction are simulated.

Recall farmer i’s demand on land is

\[
K_t^F(i) = \frac{1}{q_t - \frac{1}{R_{t}}E_{t}^{p}q_{t+1}}\left[ (a + a\epsilon_t + q_t)K_{t-1}^F(i) - R_{t}b_{t-1}^F(i) \right] \tag{98}
\]

Denote \( \hat{\vartheta}_t \) the percentage of real rate reduction and \( R \) the steady state value of interest rate. Note \( R_{t}b_{t-1}^F(i) = \frac{R + R\hat{\vartheta}_t}{R + R\hat{\vartheta}_t - 1}E_{t-1}^{p}q_{t}K_{t-1}^F \). Our model becomes

\[
(q_t - \frac{1}{R + R\hat{\vartheta}_t}(1 - \tau)E_{t}^{p}q_{t+1})K_t^F = (a + q_t - (1 - \tau)\frac{R + R\hat{\vartheta}_t}{R + R\hat{\vartheta}_t - 1}E_{t-1}^{p}q_{t})K_{t-1}^F \tag{99}
\]

\[
q_t - \frac{1}{R + R\hat{\vartheta}_t}E_{t}^{p}q_{t+1} = \frac{G'(K_t^G)}{R + R\hat{\vartheta}_t} \tag{99}
\]
Log-linearization of equation (98) and equation (99) yield

\[
\hat{K}_t^F = \frac{\frac{1-\tau}{R}}{1 - \frac{1-\tau}{R}} E_t^P \hat{q}_{t+1} - \frac{1-\tau}{1 - \frac{1-\tau}{R}} E_t^P \hat{q}_t + \hat{K}_{t-1}^F - \frac{1-\tau}{1 - \frac{1-\tau}{R}} (1 + \frac{1}{R}) \hat{\theta}_t + \frac{1-\tau}{R} \hat{\theta}_{t-1} \quad (100)
\]

and

\[
\frac{R}{R-1} (\hat{q}_t + \hat{\theta}_t - \frac{1}{R} E_t^P \hat{q}_{t+1}) = \frac{1}{\eta} \hat{K}_t^F \quad (101)
\]

Collateral holding and collateral prices depend not only on current real rate reduction to interest rate but on lagged one. We assume agents could observe \(\hat{\theta}_t\) and regressing collateral prices on constant, collateral holding and shock to interest rate. Agents’ conditional expectation is \(E_t^P \hat{q}_{t+1} = \zeta_t^m + \zeta_t^p \hat{K}_t^F + \zeta_t^v \hat{\theta}_t\) and \(E_t^P \hat{q}_t = \zeta_t^m + \zeta_t^p \hat{K}_{t-1}^F + \zeta_t^v \hat{\theta}_{t-1}\). It could be shown that the actual law of motion for collateral prices is

\[
\hat{q}_t = T_1(\zeta_t^{m-1}, \zeta_t^m, \zeta_t^v) + T_2(\zeta_t^p, \zeta_t^v) \hat{K}_{t-1}^F + T_3(\zeta_t^p, \zeta_t^v) \hat{\theta}_t + T_4(\zeta_t^p, \zeta_t^v) \hat{\theta}_{t-1} \quad (102)
\]

where

\[
T_1(\zeta_t^{m-1}, \zeta_t^m, \zeta_t^v) = \frac{(\xi_1 \zeta_t^{m-1} - \xi_2 \zeta_t^m)(1 + \frac{\zeta_t^p}{\zeta_3 R}) - \xi_1 \zeta_t^v \frac{\zeta_t^p}{\zeta_3 R}}{1 + (1 + \frac{\zeta_t^p}{\zeta_3 R}) - \xi_1 \zeta_t^v \frac{1}{\xi_3 R}} ,
\]

\[
T_2(\zeta_t^p, \zeta_t^v) = \frac{\xi_3 - \xi_2 \zeta_t^p}{1 - \frac{\xi_1 \zeta_t^v}{\xi_3 + \xi_t^v}} ,
\]

\[
T_3(\zeta_t^p, \zeta_t^v) = \frac{(1-\tau)(1-\zeta_t^{m-1})(1-\zeta_t^p)}{1 - \frac{1-\tau}{R}(1 + \zeta_t^p)} - 1 ,
\]

and

\[
T_4(\zeta_t^p, \zeta_t^v) = \frac{(1-\tau)(1-\zeta_t^{m-1})(1+\zeta_t^p)}{1 - \frac{1-\tau}{R}(1 + \zeta_t^p)} .
\]

## I The AR(1) process and its continuous-time version: Ornstein-Uhlenbeck process

The agents entertain a model that collateral price follows Ornstein-Uhlenbeck process (OUP) as following\(^{25}\):

\[
dq_t = \eta (\bar{q} - q_t) dt + \sigma dz \quad (103)
\]

where \(dz\) represents the increment of a wiener process. The OUP is governed by three parameters, namely, the long-run mean \(\bar{q}\), speed of reversion \(\eta\) and the innovation shock variance.

In financial economics theory and in the practice, for example, the price for commodities, e.g., oil, housing, are usually modeled by a mean-reversion process\(^{26}\), i.e., Ornstein-Uhlenbeck process. The justification for mean reversion process is based on microeconomic theory that the price of a commodity ought to be tied to its long-run

---

\(^{24}\)The definitions of \(\xi_i's\) are in the text.

\(^{25}\)We focus on learn about future collateral price and do not include the stochastic productivity shock in the perceived law of motion of the agents.

\(^{26}\)For example, see Dixit and Pindyck (1994)
marginal production cost. Although the price of commodities has some oscillations, it tends to revert back to a "norm" long-run equilibrium level.

Now define \( v_t \equiv e^{\nu t} q_t \). Apply Ito’s Lemma to \( v_t \)

\[
 dv_t = \frac{\partial v}{\partial q} dq_t + \frac{\partial v}{\partial t} dt
\]

\[
 = e^{\nu t} [\eta (\bar{q} - q_t) + \sigma dw_t] + \eta e^{\nu t} q_t dt
\]

\[
 = e^{\nu t} \eta d\bar{q} + e^{\nu t} \sigma dw_t
\]

Computing \( v_t \) by integration

\[
v_t = v_0 + \int_0^t dv_s
\]

\[
 = q_0 + \int_0^t e^{\nu s} \eta d\bar{q} + \int_0^t e^{\nu s} \sigma dw_s
\]

\[
 = q_0 + \bar{q} (e^{\nu t} - 1) + \int_0^t e^{\nu s} \sigma dw_s
\]

So \( q_t = q_0 e^{-\eta t} + \bar{q} (1 - e^{-\eta t}) + \sigma \int_0^t e^{\eta (s-t)} dw_s \). The asset price is distributed as

\[
q_t \sim \mathcal{N}(q_0 e^{-\eta t} + \bar{q} (1 - e^{-\eta t}), \frac{\sigma^2}{2\eta} (1 - e^{-2\eta t}))
\]

The limit is following:

\[
\lim_{t \to \infty} E[q_t] = \bar{q}
\]

\[
\lim_{t \to \infty} \text{Var}[q_t] = \frac{\sigma^2}{2\eta}
\]

We assume further that the agents don’t know the parameters of the model and need to estimate them using historical data. With data available in discrete time, the agents need to discretize their model. The above process is a continuous time version of the first-order autoregressive process in discrete time. Specifically, equation (103) is the limiting case as \( \Delta t \to 0 \) of the following AR(1) process:

\[
q_t - q_{t-1} = \bar{q} (1 - e^{-\eta}) + (e^{-\eta} - 1) q_{t-1} + \epsilon_t
\]

The correspondence between parameters in original OUP and parameters in AR(1) process are following

\[
q_t - q_{t-1} = a + bq_{t-1} + \epsilon_t
\]

where \( \eta = -\log(1 + b) \), \( \bar{q} = -\frac{a}{b} \), \( \sigma = \sigma \sqrt{\frac{2\log(1+b)}{(1+b)^2 - 1}} \).
Equivalently, agents’ perception of the asset price is
\[ q_t = a + (1 + b)q_{t-1} + \eta_t \quad (104) \]
Note a higher AR(1) coefficient is equivalent to slower speed of mean reversion.

Suppose the agents perceive asset price follows an AR(1) process.
\[ q_t = \beta^m + \beta^p q_{t-1} + \eta_t \quad (105) \]
given \( q_{-1} \) with \( \beta^m = a, \beta^p = (1 + b) \), and \( \eta_t \sim i.i.N(0, H^{-1}) \). \( H = \frac{1}{\sigma_\eta^2} \) is the precision of the shock\(^{27}\). The agents are uncertain about the mean, persistence, of asset price, i.e., \( \beta \equiv (\beta^m, \beta^p) \), and covariance matrix of innovation \( \sigma_\eta^2 \). They try to learn these parameters in light of data using Bayesian updating.

### J Proof of Proposition 10

Recall local stability of the MSV RE is determined by the stability of the following associated ODEs
\[
\begin{align*}
\frac{d\beta^m}{d\tau} &= T_1(\beta^m, \beta^p) - \beta^m \\
\frac{d\beta^p}{d\tau} &= T_2(\beta^p, \beta^p) - \beta^p
\end{align*}
\]
where \( T_1(\beta^m, \beta^p) = \frac{(z_1 - z_2)\beta^m}{1 - z_1 \beta^p} \), \( T_2(\beta^m, \beta^p) = \frac{1 - z_2 \beta^p}{1 - z_1 \beta^p} \), \( z_1 = \gamma_1 = \frac{1}{R}(1 + \frac{1}{\eta}) \), \( z_2 = \gamma_2 + \frac{1}{R} = \frac{1}{\eta} + \frac{1}{R}; \beta^m = 0 \) and \( \beta^p = \frac{\eta}{1 + \eta} \).

The E-stability condition for MSV equilibrium (53) for the economy represented by equation (52) requires the eigenvalues of the Jacobian of the right hand side is negative. Since \( \beta^m \) does not show up in the ODE for \( \beta^p \), the eigenvalues will be on the diagonal and only two partial derivatives, i.e., \( \frac{\partial T_1(\beta^m, \beta^p)}{\partial \beta^m} \bigg|_{\beta^m = \beta^m, \beta^p = \beta^p} \) and \( \frac{\partial T_2(\beta^p, \beta^p)}{\partial \beta^p} \bigg|_{\beta^p = \beta^p} \), matter for the E-stability. The first derivative
\[
\frac{\partial T_1(\beta^m, \beta^p)}{\partial \beta^m} \bigg|_{\beta^m = \beta^m, \beta^p = \beta^p} = \frac{(z_1 - z_2)}{1 - z_1 \beta^p}
\]
which is negative given \( z_1 < z_2 \) and \( 1 - z_1 \beta^p > 0 \). The second derivative is
\[
\frac{\partial T_2(\beta^m, \beta^p)}{\partial \beta^p} \bigg|_{\beta^m = \beta^m, \beta^p = \beta^p} = -\frac{z_2(1 - z_1 \beta^p) - z_1(1 - z_2 \beta^p)}{(1 - z_1 \beta^p)^2}
\]
\[= -\frac{z_2 + z_1 \beta^p}{1 - z_1 \beta^p} \]
Note in the last equality we use \( \beta^p = \frac{1 - z_2 \beta^p}{1 - z_1 \beta^p} \). The second derivative is negative because \( z_2 + z_1 \beta^p > 0 \) and \( 1 - z_1 \beta^p > 0 \).

\(^{27}\)It is conveniently working with precision which is the reciprocal of the variance.
\textbf{K Proof of Proposition 11}

Let us firstly look at part (1). The denominator $1 - \beta_t z_2 > 0$. Given $\beta^{p}_{t-1}, \beta^{p}_t$, we derive $\beta^{p}_{t+1}$

\[
T_2 (\beta^{p}_{t-1}, \beta^{p}_t) = \frac{1 - z_2 \beta^{p}_{t-1}}{1 - \beta^{p}_t z_1}
\]

\[
= \beta^p + \frac{- \beta^p (1 - \beta^p z_1) + (1 - z_2 \beta^p_{t-1})}{1 - \beta^p z_1}
\]

\[
= \beta^p + \frac{- \beta^p (1 - \beta^p z_1) + (1 - z_2 \beta^p_{t-1}) + \beta^p (1 - \beta^p z_1) - (1 - z_2 \beta^p)}{1 - \beta^p z_2}
\]

\[
= \beta^p + \frac{z_2(\beta^p - \beta^p_{t-1}) - \beta^p z_1 (\beta^p - \beta^p_t)}{1 - \beta^p z_1}
\]

In the third equality we use $\beta^p = \frac{1-z_2 \beta^p}{1-z_1 \beta^p}$, which holds for the RE value of collateral price persistence $\beta^p$.

Given $\frac{1}{\eta} + \frac{1}{R} = z_2 > z_1 = \frac{1}{\eta \rho} + \frac{1}{R}$, $\beta^p < 1$, $\beta^p_t \leq \beta^p$ and $\beta^p_t > \beta^p_{t-1}$, we obtain $T(\beta^p_{t-1}, \beta^p_t) > \beta^p > \beta^p_t$. So $\beta^p_{t+1} > \beta_t$. Similarly for part (2), given $z_2 > z_1$, $\beta^p < 1$, $\beta^p_t \geq \beta^p$ and $\beta^p_t < \beta^p_{t-1}$, we obtain $T(\beta^p_{t-1}, \beta^p_t) < \beta^p < \beta^p_t$ and $\beta^p_{t+1} < \beta^p_t$.

\textbf{L Proof of Proposition 12}

Part 1) Local stability of the MSV RE is determined by the stability of the following associated ODEs

\[
\frac{d\alpha^{m}}{d\tau} = T_1(\alpha^{m}, \alpha^{p}) - \alpha^{m}
\]

\[
\frac{d\alpha^{p}}{d\tau} = T_2(\alpha^{p}, \alpha^{p}) - \alpha^{p}
\]

where $T_1(\alpha^{m}, \alpha^{p}) = \frac{(\delta_1 - \delta_2) \alpha^{m}}{1 - \delta_1 \alpha^p}$, $T_2(\alpha^{m}, \alpha^{p}) = \frac{1 - \delta_2 \alpha^p}{1 - \delta_1 \alpha^p}$, $\delta_1 = \xi_1$, $\delta_2 = \xi_2 + \frac{1}{R} = (1 - \tau) \frac{1 - \frac{R}{R} - \frac{1}{R} (1 - \tau)}{1 - \frac{1}{R} (1 - \tau)} + \frac{1}{R}$, $\alpha^{m} = 0$, and $\alpha^{p} = \frac{1}{1 + \frac{1 - \frac{R}{R} - \frac{1}{R} (1 - \tau)}{1 - \frac{1}{R} (1 - \tau)}}$.

The E-stability condition for MSV equilibrium (62) requires the eigenvalues of the Jacobian of the right hand side is negative. Since $\alpha^{m}$ does not show up in the ODE for $\alpha^{p}$, the eigenvalues will be on the diagonal and only two partial derivatives, i.e.,

\footnote{This is assumed to hold, otherwise the subjective discounted sum of utility will be unbounded and the optimal plan is not well-defined.}
\frac{\partial T_1(\alpha_\kappa, \alpha_\rho)}{\partial \alpha_\kappa}|_{\alpha_\kappa = \pi_\kappa, \alpha_\rho = \pi_\rho}$ and $\frac{\partial T_2(\alpha_\kappa, \alpha_\rho)}{\partial \alpha_\rho}|_{\alpha_\rho = \pi_\rho}$ matter for the E-stability. The first derivative

\frac{\partial T_1(\alpha_\kappa, \alpha_\rho)}{\partial \alpha_\kappa}|_{\alpha_\kappa = \pi_\kappa, \alpha_\rho = \pi_\rho} = \frac{(\delta_1 - \delta_2)}{1 - \delta_1 \pi_\rho}

which is negative given $\delta_1 < \delta_2$ and $1 - \delta_1 \pi_\rho > 0$. The second derivative is

\frac{\partial^2 T_2(\alpha_\kappa, \alpha_\rho)}{\partial \alpha_\kappa \partial \alpha_\rho}|_{\alpha_\rho = \pi_\rho} = \frac{-\delta_2 (1 - \delta_1 \pi_\rho) - \delta_1 (1 - \delta_2 \pi_\rho)}{(1 - \delta_1 \pi_\rho)^2}

= \frac{-\delta_2 + \delta_1 \pi_\rho}{1 - \delta_1 \pi_\rho}

Note in the last equality we use $\pi_\rho = \frac{1 - \delta_1 \pi_\rho}{1 - \delta_1 \pi_\rho}$. The second derivative is negative because $\delta_2 + \delta_1 \pi_\rho > 0$ and $1 - \delta_1 \pi_\rho > 0$.

Part 2) could be proved following the proof of proposition 11 by noting $\delta_2 > \delta_1$. 

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