No. 2004/07

A No-Arbitrage Approach to Range-Based Estimation of Return Covariances and Correlations

Michael W. Brandt, Francis X. Diebold

Center for Financial Studies
an der J. W. Goethe-Universität ▪ Taunusanlage 6 ▪ D-60329 Frankfurt am Main
Telefon: (+49)069/242941-0
Fax: (+49)069/242941-77 ▪ http://www.ifk-cfs.de ▪ E-Mail: info@ifk-cfs.de

Center for Financial Studies

The *Center for Financial Studies* is a nonprofit research organization, supported by an association of more than 120 banks, insurance companies, industrial corporations and public institutions. Established in 1968 and closely affiliated with the University of Frankfurt, it provides a strong link between the financial community and academia.

The CFS Working Paper Series presents the result of scientific research on selected topics in the field of money, banking and finance. The authors were either participants in the Center´s Research Fellow Program or members of one of the Center´s Research Projects.

If you would like to know more about the *Center for Financial Studies*, please let us know of your interest.

Prof. Dr. Jan Pieter Krahnen

Prof. Volker Wieland, Ph.D.

Center for Financial Studies
an der J. W. Goethe-Universität ▪ Taunusanlage 6 ▪ D-60329 Frankfurt am Main
Telefon: (+49)069/242941-0
Fax: (+49)069/242941-77 ▪ http://www.ifk-cfs.de ▪ E-Mail: info@ifk-cfs.de

CFS Working Paper No. 2004/07

A No-Arbitrage Approach to Range-Based Estimation of Return Covariances and Correlations[♠]

Michael W. Brandt[♦] and Francis X. Diebold[♥]

First Draft: September 2001
This Draft/Print: March 27,2002

Abstract:
We extend the important idea of range-based volatility estimation to the multivariate case. In particular, we propose a range-based covariance estimator that is motivated by financial economic considerations (the absence of arbitrage), in addition to statistical considerations. We show that, unlike other univariate and multivariate volatility estimators, the range-based estimator is highly efficient yet robust to market microstructure noise arising from bid-ask bounce and asynchronous trading. Finally, we provide an empirical example illustrating the value of the high-frequency sample path information contained in the range-based estimates in a multivariate GARCH framework.

Keywords: Range-based estimation, volatility, covariance, correlation, absence of arbitrage, exchange rates, stock returns, bond returns, bid-ask bounce, asynchronous trading

[♠] This work was supported by the National Science Foundation, the Wharton Financial Institutions Center, and the Rodney L. White Center for Financial Research at the Wharton School. We thank Tim Bollerslev, Celso Brunetti, Rob Engle, Joel Hasbrouck, Peter Lildolt, Jeff Russell and Neil Shephard for insightful discussions and comments. Clara Vega provided outstanding research assistance.

[♦] Department of Finance, University of Pennsylvania, and NBER, phone: 215-898-3609, Email: brandtm@wharton.upenn.edu

[♥] Departments of Economics, Finance and Statistics, University of Pennsylvania, and NBER phone: 215-898-1507, e-mail: fdiebold@wharton.upenn.edu

1. Introduction

The price range, defined as the difference between the highest and lowest log asset prices over a fixed sampling interval (for concreteness, we focus on a one-day interval), has a long, colorful and distinguished history of use as a volatility estimator.[1] As emphasized most recently by Alizadeh, Brandt and Diebold (2002), the range has many attractive properties. First, it is a highly efficient volatility proxy. The intuition is simple. The range is a potent distillation of the valuable information about volatility contained in the entire intraday price path, whereas alternative volatility proxies based on the daily return, such as the daily squared return, use only the opening and closing prices. Hence, for example, on days when the security price fluctuates substantially throughout the day but, by chance, the closing price is close to the opening price, the daily squared return indicates low volatility despite the large intraday price fluctuation. The daily range, in contrast, reflects the intraday price fluctuations and therefore indicates correctly that the volatility is high.

Second, the range appears robust to common forms of market microstructure noise, such as bid-ask bounce. Again, the intuition is simple. In the presence of bid-ask bounce, the observed daily maximum is likely to be at the ask and hence too high by half the spread, whereas the observed minimum is likely to be at the bid and hence too low by half the spread. On average, then, bid-ask bounce inflates the range by the average spread, which is small in liquid markets.

Finally, data on the range, which is also a popular indicator in technical analysis (e.g., Edwards and Magee, 1997), are widely available for exchange-traded futures contracts (including currencies, Treasury bond, and stock indices) and individual stocks, not only presently but also, in many cases, over long historical time spans. In fact, the range has been reported for many years in major business newspapers through so-called "candlestick plots," showing the daily high, low, and close.

Despite these appealing properties of the range, one cannot help but notice a large gap in the range-based volatility estimation literature: it is entirely univariate. That is, although range-based variance estimation has been extensively discussed and refined, range-based *covariance* estimation remains uncharted territory. The reason is that it is not at all obvious how to construct an appropriate range-based covariance estimator. Intuitively plausible candidates such as the maximal daily divergence between two log prices, $\text{Range}[\ln p_1 - \ln p_2]$, make sense only when the covariance is known to be positive, and moreover, they cannot be computed without the underlying high-frequency data because

[1] The relevant literature includes Garman and Klass (1980), Parkinson (1980), Beckers (1983), Ball and Torous (1984), Rogers and Satchell (1991), Kunitomo (1992), and Yang and Zhang (2000).

$\text{Range}[\ln p_1 - \ln p_2] \neq \text{Range}[\ln p_1] - \text{Range}[\ln p_2]$. Hence the range would seem to join the ranks of other famously obvious and intuitive univariate statistics that have no similarly obvious or intuitive multivariate generalization, such as the median.

The apparent failure of range-based volatility estimation to generalize to the multivariate case is particularly unfortunate, because financial economics is intimately concerned with multivariate interactions. Consider, for example, three pillars of modern finance: asset pricing, asset allocation, and risk management. An asset's price is determined by its systematic risk, which is related to its *covariance* with the market or other risk factors. Similarly, optimal portfolio shares depend on the variances and *covariances* of asset returns, as does portfolio value at risk.

In this paper, we propose a simple and intuitive multivariate range-based volatility estimator. Our approach relies in an appealing way on financial economic considerations, in addition to statistical considerations; in particular, we invoke the absence of arbitrage opportunities. We proceed as follows. In Section 2, we begin by discussing salient aspects of univariate range-based volatility estimation, which sets the stage for the multivariate discussion to which we then turn. In particular, we show how no-arbitrage conditions can be used to express covariances in terms of variances, which may of course be estimated by standard range-based methods, in foreign exchange, fixed income, and equity contexts. We also conjecture that the appealing properties of univariate range-based volatility (reasonably high if not theoretically maximal efficiency, combined with robustness to market microstructure noise) documented by Alizadeh, Brandt and Diebold (2002) carry over to the multivariate case.

We confirm this conjecture in Section 3, in which we provide a Monte Carlo investigation of the properties of the range-based volatility and covariance estimators, and we compare them to those of realized volatilities and covariances, which have received substantial attention recently.[2] We first establish the absolute and relative performance of the two approaches in an ideal situation with no microstructure noise, after which we consider separately the effects of bid-ask bounce and asynchronous trading on the estimators. Our Monte Carlo results demonstrate that, although the range-based estimates are less efficient than the realized estimates under the abstract conditions ideal for the realized estimates, they are nevertheless quite efficient, and they are much less biased than naively-constructed realized

[2] See, for example, the review in Andersen, Bollerslev and Diebold (2002).

estimates in the presence of realistic market microstructure contamination of the data.[3] The balance struck by the range-based volatility and covariance estimators makes them potentially attractive for applied work.

In section 4, we then provide an empirical example that illustrates the value of the high-frequency sample path information contained in the range-based volatility and covariance estimates in a standard multivariate GARCH framework. In particular, we construct range-based estimates of the covariance matrix of Deutsche mark/U.S. dollar and Japanese Yen/U.S. dollar futures exchange rate returns. To evaluate the information content in the range-based variance and covariance estimates, relative to daily squared returns and return cross-products, we include the lagged range-based estimates as volatility predictors in an otherwise-standard multivariate GARCH model. We find that the range-based estimates drive out the lagged squared returns and return cross-products. Besides attesting to the fact that the range-based estimates are highly informative, the empirical example demonstrates how easy it is to incorporate range-based volatility estimation in standard empirical work.

Finally, we conclude and discuss directions for future research in Section 5.

2. Theoretical Framework

2.1. The Univariate Case

Before considering the estimation of multivariate covariances and correlations, we set the stage by considering the estimation of univariate variances. Consider a univariate stochastic volatility diffusion for the log of an asset price p_t with instantaneous volatility σ_t. Suppose we sample this process discretely at m regular times throughout the day, which lasts from time t to $t+1$, to obtain the intraday returns $r_{(m),t+k/m} = p_{t+k/m} - p_{t+(k-1)/m}$, for $k=1,...,m$. Under general conditions given in Andersen, Bollerslev, Diebold and Labys (2001a, 2001b) and Andersen, Bollerslev and Diebold (2002), the variance of these discrete-time returns over the one-day interval conditional on the sample path $\{\sigma_{t+\tau}\}_{\tau=0}^{1}$ is:

[3] By "naively-constructed," we refer to realized estimates constructed without regard for mitigating the effects of microstructure noise. Much of the applied realized volatility literature, of course, *does* attempt to control for microstructure noise in a variety of ways. Work such as Andersen, Bollerslev, Diebold and Labys (2001b), for example, attempts to insure against microstructure effects by focusing only on the deepest, most liquid markets and by working with an underlying sampling frequency much less high then the maximum obtainable. More recent work such as Zumbach, Corsi and Trapletti (2002), moreover, develops formal methods for reducing the effects of microstructure noise either by smoothing the ultra-high-frequency data before constructing realized volatilities, or by smoothing the naive realized volatilities constructed from the raw ultra-high-frequency data.

$$\bar{\sigma}_t^2 = \int_0^1 \sigma_{t+\tau}^2 \, d\tau. \tag{1}$$

The *integrated volatility* $\bar{\sigma}_t$ thus provides a canonical and natural measure of return volatility, and it features prominently in the financial economics literature (e.g., Hull and White, 1987). Because the integrated volatility is inherently unobservable, several estimators have been proposed, including estimators based on the intraday returns, daily returns, and the daily range. Let us consider them in turn.

The most intuitive volatility estimator is the realized volatility, which is simply the square-root of the sum of the intraday squared returns, used for example in Merton (1980), French, Schwert and Stambaugh (1987), Andersen, Bollerslev, Diebold, and Ebens (2001) and Andersen, Bollerslev, Diebold and Labys (2001a, 2001b). The daily realized volatility converges, in the sampling frequency *m,* to the daily integrated volatility; that is, if we define the daily realized variance estimator as:

$$\bar{\sigma}_t^2(m) = \sum_{k=1,\ldots,m} r_{(m),t+k/m}^2, \tag{2}$$

then, under regularity conditions, $\text{plim}_{m\to\infty} \bar{\sigma}_t^2(m) = \bar{\sigma}_t^2$. Hence, one can approximate, at least in principle, the daily integrated variance arbitrarily well by summing sufficiently finely sampled intraday squared returns.

An extreme, yet very popular, version of the realized volatility is the daily squared return, which is simply the daily realized variance with the intraday sampling frequency *m* set to one. Although the squared return is still an unbiased estimator of the integrated variance, it is contaminated by substantial measurement error due to the fact that it only incorporates the opening and closing prices. To gain intuition for the source of this measurement error, consider a day when the security price fluctuates substantially throughout the day but, by chance, the closing price is close to the opening price. On such a day, the squared return indicates low volatility despite the large intraday price fluctuation, and the measurement error contained in the squared return as an estimator of the integrated variance is of the same magnitude as the integrated variance itself. As a way to quantify the average magnitude of the measurement errors, Andersen and Bollerslev (1998) simulate a stochastic volatility model with empirically realistic parameters and compare the day-to-day fluctuation of the integrated variance to the day-to-day fluctuation of the squared daily return. They find that the variance of the measurement error in squared daily returns is more than twenty times the variance of the true underlying integrated variance. As a result of this overwhelming amount of measurement error, squared returns are typically smoothed, as in GARCH models, to help reduce the measurement error.

Now consider Parkinson's (1980) range-based estimator of the daily integrated variance:

$$\gamma \ = \ 0.361 \ \text{Range}_t^2. \tag{3}$$

Ideally, $\text{Range}_t = \max[\{\ln p_{t+\tau}\}_{\tau=0}^{1}] - \min[\{\ln p_{t+\tau}\}_{\tau=0}^{1}]$ (i.e., the max and min are taken over the daily sample path) but in practice the range can only be computed for a discretely sampled price process $\text{Range}_t(m) = \max[\{\ln p_{t+k/m}\}_{k=1,\ldots,m}] - \min[\{\ln p_{t+k/m}\}_{k=1,\ldots,m}]$. Like the daily squared return, the daily range-based volatility estimator evaluated at the true range is unbiased for integrated volatility, but, importantly, range-based volatility is much less contaminated by measurement error. For example, in the example we gave above to illustrate the source of the measurement error in the daily squared returns, the range accurately reflects the intraday price fluctuations and therefore indicates correctly that the volatility is high. Furthermore, Anderson and Bollerslev's (1998) simulation study reveals that, if the daily squared return is as efficient as realized volatility computed using a sampling rate of once a day (which is true by definition), then range-based volatility is approximately as efficient as realized volatility computed using a sampling rate of roughly ten times per day.

Although the range is highly efficient relative to the daily squared return, it is nevertheless less efficient than realized volatility. In particular, realized volatility computed with a sufficiently high sampling rate theoretically dominates all other estimators because it becomes free of measurement error as $m \to \infty$. Against the theoretical superiority of realized volatility under ideal conditions – "Merton's utopia" in the colorful language of Bai, Russell and Tiao (2000) – one must balance a number of real-world complications that may cause difficulty for naively-constructed realized volatilities, but not for range-based volatilities. First, the sampling rate *m* can not literally be taken to infinity as required for the superiority of realized volatility because the sampling frequency is ultimately constrained by transactions time, which is discrete. Second, the ultra-high frequency data required for theoretically ultra-efficient realized volatilities is often unavailable at present, and certainly unavailable historically, whereas the daily range is broadly disseminated and is often available historically. Finally, as shown by Alizadeh, Brandt and Diebold (2002), naively-constructed realized volatilities can be severely biased due to bid-ask bounce, whereas range-based volatilities are robust to this form of market microstructure noise. The intuition is that in the presence of a bid-ask spread the observed price is a noisy version of the true price because it equals the true price plus or minus half the spread, depending on whether a trade is buyer- or seller-initiated. Because transactions tend to bounce between buys and sells, the average size of the squared intraday returns is increased. By summing the squared intraday returns, each of which is biased upward, the realized volatility contains a *cumulated* and therefore potentially large bias, which becomes more severe as returns are sampled more frequently. The range, in contrast, is less likely to be seriously contaminated by bid-ask bounce. The highest observed price is likely to be at the ask and hence too high by half the spread, whereas the lowest observed price is likely to be at the bid and hence too low by half the spread. On average, the range is inflated only by the average spread, which is small in liquid markets.[4]

In summary, range-based volatilities achieve a potentially attractive compromise: they are convenient to use (both from a data collection and computational perspective), they perform well under theoretical conditions ideal for realized volatility (in particular, they are much more efficient than daily squared returns), and they may perform better than realized volatilities in the presence of real-world complications due to market microstructure.

2.2. The Multivariate Case

We can now consider the multivariate case, which, as we stressed in the introduction, is of crucial importance for finance. Let σ_{ij}^2 denote the *ij*-th element of the multivariate diffusion matrix Σ. Then, the *ij*-th daily integrated covariance:

$$\bar{\sigma}_{ij,t}^2 = \int_0^1 \sigma_{ij,t+\tau}^2 d\tau \tag{4}$$

can be consistently estimated by the daily realized covariance:

$$\bar{\sigma}_{ij,t}^2(m) = \sum_{k=1,\ldots,m} r_{i,(m),t+k/m} \times r_{j,(m),t+k/m}. \tag{5}$$

In precise parallel to the univariate case, we have, under regularity conditions, $\text{plim}_{m\to\infty} \bar{\sigma}_{ij,t}^2(m) = \bar{\sigma}_{ij,t}^2$.[5] Hence, one can approximate, at least in principle, the daily integrated covariance arbitrarily well by summing sufficiently finely sampled intraday return cross products.

The attractive blend of convenience, efficiency and robustness achieved by the range-based

[4] Moreover, one could readily perform a bias correction by subtraction the average spread from the range. We thank Joel Hasbrouck for this observation.

[5] See Andersen, Bollerslev, Diebold and Labys (2001a, 2001b) and Andersen, Bollerslev and Diebold (2002).

volatility estimator in the univariate case makes one eager for extension to the multivariate case. We now proceed to do so. The basic idea is very simple, and the implementation varies slightly depending on whether the application is to foreign exchange, bonds or stocks. We consider each in turn.

2.2.1. Foreign Exchange

In foreign exchange markets, absence of triangular arbitrage implies a deterministic relationship between any pair of dollar rates and the corresponding cross rate. Consider two dollar exchange rates, denoted *A/$* and *B/$*. Then, in the absence of triangular arbitrage, the cross-rate is $A/B = (A/\$)/(B/\$)$ and hence the continuously compounded *A/B* return is:

$$\Delta\ln A/B \;=\; \Delta\ln A/\$ \;-\; \Delta\ln B/\$. \qquad (6)$$

Taking variances yields:

$$\mathrm{Var}[\Delta\ln A/B] \;=\; \mathrm{Var}[\Delta\ln A/\$] \;+\; \mathrm{Var}[\Delta\ln B/\$] \;-\; 2\,\mathrm{Cov}[\Delta\ln A/\$, \Delta\ln B/\$], \qquad (7)$$

or

$$\mathrm{Cov}[\Delta\ln A/\$, \Delta\ln B/\$] \;=\; \frac{1}{2}\Big(\mathrm{Var}[\Delta\ln A/\$] \;+\; \mathrm{Var}[\Delta\ln B/\$] \;-\; \mathrm{Var}[\Delta\ln A/B]\Big). \qquad (8)$$

This suggests a natural range-based return covariance estimator:

$$\hat{\mathrm{Cov}}[\Delta\ln A/\$, \Delta\ln B/\$] \;=\; \frac{1}{2}\Big(\hat{\mathrm{Var}}[\Delta\ln A/\$] \;+\; \hat{\mathrm{Var}}[\Delta\ln B/\$] \;-\; \hat{\mathrm{Var}}[\Delta\ln A/B]\Big), \qquad (9)$$

where $\hat{\mathrm{Var}}[\Delta\,\cdot]$ is Parkinson's (1980) range-based return variance estimator. We then assemble the estimated variance-covariance matrix as:

$$\hat{\Sigma} \;=\; \begin{bmatrix} \hat{\mathrm{Var}}[\Delta\ln A/\$] & \hat{\mathrm{Cov}}[\Delta\ln A/\$, \Delta\ln B/\$] \\ \hat{\mathrm{Cov}}[\Delta\ln A/\$, \Delta\ln B/\$] & \hat{\mathrm{Var}}[\Delta\ln B/\$] \end{bmatrix}. \qquad (10)$$

In higher dimensional cases, we proceed in precisely analogous fashion, estimating the variances using Parkinson's method and estimating each pairwise covariance as above, and then assembling the results into an estimated covariance matrix.

2.2.2. Fixed Income

In fixed income markets, the absence of arbitrage implies a deterministic relationship among any two zero-coupon bond prices and the corresponding forward contract. Specifically, consider two bonds with maturities T_1 and T_2 and prices $P(T_1)$ and $P(T_2)$, with $T_1 < T_2$. The price of a forward contract between times T_1 and T_2 is then $F(T_1,T_2) = P(T_2) / P(T_1)$. Taking logs, $f(T_1,T_2) = p(T_2) - p(T_1)$, first differences, $r_f(T_1,T_2) = r(T_2) - r(T_1)$, and finally variances yields:

$$\mathrm{Var}[r_f(T_1,T_2)] = \mathrm{Var}[r(T_2)] + \mathrm{Var}[r(T_1)] - 2\,\mathrm{Cov}[r(T_1), r(T_2)], \tag{11}$$

or

$$\mathrm{Cov}[r(T_1), r(T_2)] = \frac{1}{2}\Big(\mathrm{Var}[r(T_2)] + \mathrm{Var}[r(T_1)] - \mathrm{Var}[r_f(T_1,T_2)]\Big). \tag{12}$$

Hence we form the covariance estimator:

$$\hat{\mathrm{Cov}}[r(T_1, r(T_2)] = \frac{1}{2}\Big(\hat{\mathrm{Var}}[r(T_2)] + \hat{\mathrm{Var}}[r(T_1)] - \hat{\mathrm{Var}}[r_f(T_1,T_2)]\Big) \tag{13}$$

and assemble the estimated variance-covariance matrix precisely as in the foreign exchange case:

$$\hat{\Sigma} = \begin{bmatrix} \hat{\mathrm{Var}}[r(T_1)]) & \hat{\mathrm{Cov}}[r(T_1), r(T_2)] \\ \hat{\mathrm{Cov}}[r(T_1), r(T_2)] & \hat{\mathrm{Var}}[r(T_2)] \end{bmatrix}. \tag{14}$$

Extension to higher-dimensional cases is straightforward.

2.2.3. Equities

The return on a two-equity portfolio with shares λ and $1-\lambda$, denoted $r_p = \lambda r_1 + (1-\lambda) r_2$, has a variance of:

$$\mathrm{Var}[r_p] = \lambda^2 \mathrm{Var}[r_1] + (1-\lambda)^2 \mathrm{Var}[r_2] + 2\lambda(1-\lambda)\,\mathrm{Cov}[r_1, r_2], \tag{15}$$

so that:

$$\mathrm{Cov}[r_1, r_2] = \frac{1}{2\lambda(1-\lambda)}\Big(\mathrm{Var}[r_p] - \lambda^2 \mathrm{Var}[r_1] - (1-\lambda)^2 \mathrm{Var}[r_2]\Big). \tag{16}$$

This suggests the covariance estimator:

$$\hat{\text{Cov}}[r_1,r_2] = \frac{1}{2\lambda(1-\lambda)}\left(\hat{\text{Var}}[r_p]-\lambda^2\hat{\text{Var}}[r_1]-(1-\lambda)^2\hat{\text{Var}}[r_2]\right) \tag{17}$$

and the corresponding covariance matrix estimator:

$$\hat{\Sigma} = \begin{bmatrix} \hat{\text{Var}}[r_1] & \hat{\text{Cov}}[r_1,r_2] \\ \hat{\text{Cov}}[r_1,r_2] & \hat{\text{Var}}[r_2] \end{bmatrix}. \tag{18}$$

Extension to higher-dimensional cases is again straightforward. Note also that this method of estimating the covariance via the range of the two-asset portfolio return is generally applicable to any two assets – not just equities – so long as data on the portfolio return range is available.

2.2.4. Discussion

Our no-arbitrage approach to range-based covariance estimation is widely applicable in the foreign exchange context because daily ranges of all legs of many currency triangles are available.[6] In the fixed income context, it is applicable to select maturities for which liquid bonds are aligned with liquid forward or futures contracts, such as the three- and six-month Treasury bills and the three-month Treasury bill futures or the five- and ten-year Treasury notes and the five-year Treasury note futures. In the equity context, however, the direct applicability of our approach is limited by the fact that historical data on the range of two-asset portfolios are typically not available (notable exceptions are the TSE 100, TSE 200, and TSE 300 indices of the Toronto stock exchange and the ASX 100, ASX 200, and ASX 300 indices of the Australian stock exchange). It is worth noting, however, that the increasing availability of high-frequency return data makes feasible the direct calculation of the two-asset portfolio return range. Although one can in this case bypass the range and immediately calculate realized variances and covariances, the robustness of the range to market microstructure noise, illustrated in Alizadeh, Brandt and Diebold (2002) and in the next section of this paper, makes the range an attractive volatility proxy as well.

[6] For example, Datastream provides as much as 40 years of historical data on the daily high, low, and closing prices of 37 British pound denominated currencies and of 14 Swiss franc denominated currencies. The International Monetary Market (IMM), a subsidy of the Chicago Mercantile Exchange (CME), recently introduced futures and options contracts on Euro/British pound, Euro/Swiss franc, and Euro/Japanese yen cross rates. Finally, the New York Board of Trade (NYBOT) offers futures contracts on 14 cross-currencies, including seven Euro denominated contracts.

We conjecture that just as market microstructure noise may corrupt realized variances but not range-based variances, so too may it corrupt naively-constructed realized covariances but not range-based covariances. In particular, it is natural to suspect that asynchronous trading biases realized covariances because the estimator requires identical timing of the pairs of returns used to form the cross-products. Range-based covariances, in contrast, avoid this problem by taking an indirect route based on the absence of arbitrage. We provide a detailed assessment of this conjecture in the next section.

Thus far we have said little about the theoretical properties of the range-based covariance estimator. One obvious point is that the estimator is unbiased under the same conditions that deliver unbiasedness of Parkinson's (1980) variance estimator because it is a linear combination of variances. But any such claim under ideal conditions is likely invalid in the presence of market microstructure noise, which is a crucial aspect of financial market reality.[7] Hence we will not dwell on theoretical properties likely to be irrelevant in practice.

A similarly obvious and related point is that $\hat{\Sigma}$ is in general not guaranteed to be positive definite, and moreover, even if conditions could be established under which positive semi-definiteness is guaranteed in principle, it would not be guaranteed in practice because any feasible estimator is based on data that is contaminated by market microstructure noise. We impose positive definiteness by estimating the Cholesky factor P of Σ, rather than Σ itself, where P is a unique lower-triangular matrix defined by $\Sigma = PP'$. Note that the elements of P are functions of the elements of Σ. Hence we insert our range-based estimators of the relevant variances and covariances into P (computed analytically) to obtain an estimator of the Cholesky factor $\hat{P}$ and then form the estimator $\hat{P}\hat{P}'$ of the covariance matrix. Because the estimated Cholesky factor $\hat{P}$ will be complex when $\hat{\Sigma}$ is not positive definite, we define $\hat{P}'$ as the conjugate transpose, which guarantees that $\hat{P}\hat{P}'$ is real.[8]

Ultimately, then, the interesting questions for financial economists center not on the theoretical

---

[7] Moreover, even in the absence of microstructure noise, bias may creep in. First, nonlinear transformations, such as conversion from covariances to correlations, obviously inject bias. Second, the Parkinson estimator assumes continuous sampling of an underlying diffusion, whereas in practice sampling is necessarily discrete because it can be made no finer than transaction-by-transaction. This introduces a downward bias in the Parkinson volatility estimator because the observed range is only a lower bound for the true range. Rogers and Satchell (1991) propose a bias correction, but the bias is negligible for range-based estimators constructed using ranges based on the very high – even if not infinite – frequency sampling relevant in practice. Hence we do not pursue bias corrections in this paper.

[8] Other ways to guarantee positive definiteness include the shrinkage approach of Ledoit and Wolf (2001) or the perturbation methods of Gill, Murray and Wright (1981) or Schnabel and Eskow (1999).

properties of range-based covariance and correlation estimates under abstract conditions surely violated in practice, but rather on their performance in realistic situations involving small samples, discrete sampling, and market microstructure noise. As we argued above, we have reason to suspect good performance of the range-based approach, both because of its high efficiency due to the use of the information in the intraday sample path, and because of its robustness to microstructure noise. We now turn to a Monte Carlo analysis designed to illuminate precisely those issues.

3. Monte Carlo Exploration

3.1. Ideal Conditions

We initially ignore market microstructure issues. We assume that two dollar-denominated exchange rates P_1 and P_2 evolve as driftless diffusions with annualized volatilities σ of 15 percent, a covariance of 0.9, and hence a correlation ρ of 0.4. We further assume that at each instant the cross-rate P_3 is determined by the absence of triangular arbitrage as the ratio of the two dollar rates. Starting at $P_{1,0}= P_{2,0}= 1$, we simulate 24 hours worth of m regularly spaced intraday log price observations using:

$$p_{i,t+k/m}=p_{i,t+(k-1)/m}+\sigma\sqrt{250/m}\,\varepsilon_{i,t+k/m},\ \text{ for }\ i=\{1,2\},\ \text{ and }\ p_{3,t}=p_{1,t}-p_{2,t}, \tag{19}$$

where $p_i=\ln P_i$, $[\varepsilon_1,\varepsilon_2]$ are standard normal innovations with correlation ρ, and there are 250 trading days per year. We consider sampling frequencies m ranging from m=4 (one observation every six hours) to m=1440 (one observation every minute) and use the resulting data to compute the daily range and intraday returns. We then construct three estimates of the volatilities, covariance, and correlation of the two dollar rates. Specifically, we construct range-based covariance matrix estimates using Parkinson's variance estimator (3) and equation (9), and, for comparison, we compute the realized covariance matrix using two different approaches. First, in parallel fashion to the range-based estimator, we use the three realized variances to construct an estimate of the covariance. Second, we compute the realized covariance directly using the cross-products of intraday returns, as in equation (5). We repeat this procedure 10,000 times and report the means, standard deviations, and root mean squared errors of the resulting sampling distributions in Table 1.

The results for the volatilities are familiar from Alizadeh, Brandt, and Diebold (2002).[9] The range-based estimates are downward biased because the range of the discretely sampled process is strictly less than the range of the underlying diffusion, as discussed earlier. The magnitude of the bias decreases

[9] Since p_1 and p_2 follow the same stochastic process, we analyze only the volatility estimates for p_1.

as the sampling frequency increases. But, even in the limit as $m \to \infty$, the range is still only a noisy volatility proxy, which means that the standard deviation and RMSE of the range-based volatility estimator settle down to non-zero values. The realized volatility behaves quite differently because it converges not only in expectation but also in realization to the true volatility. The more frequently the underlying diffusion is sampled, the more precise the realized volatility gets, until, in the limit, the standard deviation and RMSE of the estimator are zero. Consistent with the results of Andersen and Bollerslev (1998), sampling the diffusion every three to six hours leads to a realized volatility estimator that is comparable to the limit of the range-based estimator.

The results for the range-based covariance estimates follow from the properties of the range-based volatility estimates. The estimator $\mathrm{Cov}[\Delta p_1, \Delta p_2] = 1/2(\mathrm{Var}[\Delta p_1] + \mathrm{Var}[\Delta p_2] - \mathrm{Var}[\Delta p_3])$ involves three volatility estimates, each of which is downward biased by an amount that depends on the level of volatility (the higher the volatility the more likely that the true extremes are far from the observed extremes). Because the variance of p_3 is less than the variance of p_1 and p_2 due to the positive covariance, the covariance estimates are also downward biased because the downward bias of $\mathrm{Var}[\Delta p_1] + \mathrm{Var}[\Delta p_2]$ dominates the upward bias of $-\mathrm{Var}[\Delta p_3]$. As with the volatility estimates, the bias vanishes as we increase the sampling frequency, and the standard deviation and RMSE stabilize. The realized covariances, computed either through the no-arbitrage condition or with return cross-products, yield identical estimates that inherit the outstanding properties of the realized volatility estimates.

Finally, the range-based correlation is downward biased, although, by construction, the covariance in the numerator is *less* down-ward biased than the product of volatilities in the denominator (the correlation evaluated at the average covariance and volatilities with $m = 1440$ is 0.4336). The source of this bias is the sampling variation of the covariance and volatility estimates through Jensen's inequality. Because the sampling variation does not vanish as $m \to \infty$, the range-based correlation estimator remains downward biased even in the limit. The realized correlation does not suffer from this bias because of the convergences of the realized covariance and volatilities.

3.2. Bid-Ask Bounce

Bid-ask bounce is a well-known reality of financial market data. To examine its effect on the covariance and correlation estimates, we augment the Monte Carlo experiment with a simple model of bid-ask bounce and price discreteness taken from Hasbrouck (1999b). Specifically, we take the dollar rates from the original experiment as the true prices and compute the bid and ask quotes $B_{i,t} = \mathrm{floor}[P_{i,t} - 1/2\,\mathrm{spread}, \mathrm{tick}]$ and $A_{i,t} = \mathrm{ceiling}[P_{i,t} + 1/2\,\mathrm{spread}, \mathrm{tick}]$, where $\mathrm{floor}[x, \mathrm{tick}]$ and ceiling[x,tick] are functions that round x down or up to the nearest tick, respectively. For the cross rate, we compute the bid and ask quotes by imposing no-arbitrage given the bid and ask quotes of the dollar rates. We then take the observed prices as $P_{i,t}^{\text{obs}} = q_{i,t} B_{i,t} + (1 - q_{i,t}) A_{i,t}$, where $q_{i,t} \sim \text{Bernoulli}[1/2]$. To capture the fact that the two base currencies are denominated in dollars, which means that the sale or purchase of the dollar might involve a *simultaneous* purchase or sale of the two currencies, we allow the buy-sell indicators $q_{1,t}$ and $q_{2,t}$ to be correlated with $\text{Corr}[q_{1,t}, q_{2,t}] = \eta$. The indicator $q_{3,t}$ is independent.

Table 2 presents the results for a bid-ask spread of 0.0005 and a tick size of 0.0001, which are realistic values for currencies (see Hasbrouck, 1999b). In Panel A the correlation η is set to zero and in Panels B and C the correlation is 0.5 and 0.75, respectively. The effect of bid-ask bounce on the range-based estimates is relatively minor. In contrast, the effect on the realized volatilities, covariance, and correlation is striking. Consistent with the intuition outlined above, the realized volatilities are upward biased when the data is sampled more frequently than once every three hours. By the time the data is sampled every minute, the bias inflates the true volatility by almost 100 percent (an average estimate of 29.7 percent as opposed to a true volatility of 15 percent). The results for the realized covariance depend on whether we construct the estimator using the no-arbitrage condition or return cross-products and on the correlation of the bid-ask indicators. If we use the no-arbitrage condition, the realized covariance inherits the biases of the realized volatilities, to the point where for five-minute sampling the average estimate is *negative*. In contrast, if we use return cross-products *and* if the bid-ask indicators are independent (in Panel A), the realized covariance is unbiased. The reason is that if the bid-ask indicators are independent, then the expectation of the product of observed returns is equal to the expectation of the product of true returns. The bid-ask bounce therefore only increases the variability of the estimator. However, if the bid-ask indicators are correlated (in Panels B and C), this argument no longer holds and the realized covariance is severely positively biased because each cross-product of returns contains an upward bias due to the common component of the bid-ask indicators. Finally, the realized correlation, computed from the biased realized volatilities and biased covariance, is unreliable, ranging from -0.89 to 0.66.

3.3. Asynchronous Trading

Asynchronous trading is another market microstructure effect that is likely to affect differently the range-based and realized covariance and correlation estimates. With infrequent trading, a security has a latent true price that is only revealed when a trade occurs. Between trades, the observed price is stale at the last traded price and therefore does not reflect the true price. In a univariate setting, infrequent trading induces positive serial-correlation in the intraday returns, which, in turn, causes a downward bias in the realized volatility. In a bivariate setting, *asynchronous* infrequent trading, when the trades for the two assets do not take place at the same time, also creates a misalignment of the return-cross products that may lead to a downward bias of the realized covariance.

To capture the effect of asynchronous infrequent trading in our Monte Carlo experiment, we use the discretization (19) with $m = 17280$ (one observation per second) to simulate the latent "true" price processes. We then assign for each process n trade times randomly throughout the day and construct stale price processes for which the price is equal to the price at the previous trade time until it is reset to the latent true price at the next trade time. Hence the true prices look like continuous diffusions while the stale prices look like discrete steps that occur at different times for the different currencies. Finally, we sample these stale price processes at a regular frequency m ranging again from four to 1440 and proceed just as in Section 3.1 (i.e., there is no bid-ask bounce in this experiment).

We present the results for $n = 288$ (an average of one trade every five minutes) and $n = 1440$ (an average of one trade every minute) in panels A and B of Table 3. The range-based estimates are slightly downward biased because the infrequent trading magnifies the discretization bias. The realized volatilities are slightly downward biased due to the positive serial correlation induced by infrequent trading. Finally, when we compute the realized covariance and correlation using the no-arbitrage condition, the estimates inherit only the slight bias from infrequent trading, but when we instead use return cross-products, the estimates are severely downward biased. For example, with one-minute sampling the average realized covariance and correlation computed with return cross-products are close to zero in both panels. This extreme bias is due to the asynchronous price revelation.

3.4. Summary

The results in Table 1 confirm the theoretical fact that under conditions ideal for realized volatility estimation, all other estimators, including estimators based on the range, are suboptimal. However, range-based estimation is not much inferior. The range-based volatilities, covariances and correlations are about as efficient as their realized counterparts obtained by sampling the intraday price process four to six times per day, which is impressive given the simplicity (both from a data collection and computational perspective) of range-based estimation.

More importantly, however, tables 2 and 3 illustrates clearly that this theoretical ranking is reversed in situations of practical relevance involving realistic market microstructure noise. In the presence of either bid-ask bounce or asynchronous trading, both of which are realities of financial markets data, realized volatilities, covariances and correlations may become biased. The range-based estimates, in contrast, appear robust to these forms of market microstructure noise.

4. Empirical Example

We now use the range-based estimation approach to explore the joint volatility dynamics of the U.S. dollar prices of the Deutsche mark (recently replaced by the Euro) and Japanese yen. The mark (Euro) and yen are the two most actively traded dollar denominated currencies and their joint volatility dynamics obviously play a crucial role in international portfolio allocation and risk management.

4.1 Data and Unsmoothed Range-Based Estimates

We use daily high, low, and closing (3pm EST) prices of the Deutsche mark and Japanese yen currency futures contracts traded on the International Monetary Market (IMM), a subsidiary of the Chicago Mercantile Exchange (CME), from January 3, 1990 through December 30, 1998 (2,189 observations). A currency futures contract represents delivery of the currency on the second Wednesday of the following March, June, September, or December. Each day there are at least three futures contracts with different quarterly delivery dates traded on each currency. We use prices of the front-month contract, the contract closest to delivery and with at least ten days to delivery, which is typically the most actively traded.

Although futures on the Euro/yen cross-rate are now actively traded at the IMM and New York Board of Trade (NYBT), historical data on the mark/yen cross-rate is only available for the spot market. We prefer working with futures, as opposed to spot, exchange rates for several reasons. First, all futures prices (including the daily high and low) result from open outcry, so that all transactions are open to the market and orders are filled at the best price. Currency spot market trading, in contrast, is based on bilateral negotiation between banks, and any particular executed price is not necessarily representative of overall market conditions. Second, the closing, or "settlement," futures price is based on the best sentiment of the market at the time of close (3pm EST, after which spot market trading declines) and is widely scrutinized, because it is used for marking to market all account balances. Therefore, the futures closing price is likely to be a very accurate measure of the "true" market price at that time. Finally, futures returns are the actual returns from investing in a foreign currency, whereas spot "returns" are less meaningful unless one accounts for the interest rate differential between the two countries.

We therefore construct artificial historical data on the cross-rate futures using transactions data on the two dollar rate futures. In particular, for each day in the sample, we use the futures transactions to obtain a minute-by-minute price series for the dollar rates, linearly interpolating between transactions when necessary, and then impose no triangular arbitrage to obtain a minute-by-minute price series for the cross-rate. Finally, we take the implied daily high and low cross-rates to compute the cross-rate range.

The first two plots in Figure 1 present the Parkinson estimates of the mark and yen return volatility. The third plot shows the range-based estimates of the return correlation, obtained using the range-based variance estimates and equation (9).

4.2. Smoothed Range-Based Estimates

Although the range-based volatilities are much less contaminated by measurement error than daily squared returns, Figure 1 reveals that they are nevertheless quite noisy. It is possible, therefore, that the measurement error in range-based volatilities and correlations can be beneficially reduced by smoothing, just as, for example, the much more noisy squared returns are smoothed by GARCH models. This suggests an intriguing possibility. By including lagged range-based estimates in an otherwise-standard GARCH model, we can potentially attain the best of both worlds: use of the range incorporates valuable intraday price path information, and use of the smoothing implicit in the GARCH framework reduces the measurement error. In fact, it is natural to conjecture that the inclusion of the lagged range-based estimates will reduce, or even eliminate, the need for inclusion of the much more noisy lagged squared returns. The truth of this conjecture is an empirical matter, to which we now turn.

Consider the following bivariate diagonal GARCH(1,1) model for the daily log returns:

$$\begin{bmatrix}\Delta p_{1,t+1}\\ \Delta p_{2,t+1}\end{bmatrix} = \begin{bmatrix}\mu_1\\ \mu_2\end{bmatrix} + \begin{bmatrix}\varepsilon_{1,t+1}\\ \varepsilon_{2,t+1}\end{bmatrix} \quad \text{with} \quad \mathrm{Var}_t\begin{bmatrix}\varepsilon_{1,t+1}\\ \varepsilon_{2,t+1}\end{bmatrix} = \begin{bmatrix}h_{1,t}^2 & h_{12,t}\\ h_{12,t} & h_{2,t}^2\end{bmatrix} \tag{20}$$

and the volatility dynamics:

$$\begin{aligned} h_{1,t}^2 &= \omega_1 + \beta_1 h_{1,t-1}^2 + \alpha_1 \varepsilon_{1,t-1}^2 + \delta_1 x_{1,t-1}^2 \\ h_{2,t}^2 &= \omega_2 + \beta_2 h_{2,t-1}^2 + \alpha_2 \varepsilon_{2,t-1}^2 + \delta_2 x_{2,t-1}^2 \\ h_{12,t} &= \omega_3 + \beta_3 h_{12,t-1} + \alpha_3 \varepsilon_{1,t-1}\varepsilon_{2,t-1} + \delta_3 x_{12,t-1}, \end{aligned} \tag{21}$$

where Δp_1 is the mark/dollar return, Δp_2 is the yen/dollar return, x_1, x_2, and x_{12} are the range-based mark and yen return volatilities and covariance, and the return innovations are conditionally Gaussian.

The model is entirely standard, with one twist: we allow for the appearance of the lagged range-based variances and covariance estimates in the variances and covariance equations. The standard bivariate diagonal GARCH(1,1) model corresponds to $\delta_1 = \delta_2 = \delta_3 = 0$. In column A of Table 4, we report estimates of the model that impose this constraint. The estimates display a familiar pattern: the β

estimates are large, the α estimates are smaller, all are highly statistically significant, and in each equation the sum of the α and β estimates is close to, but less than, unity.

We now augment the GARCH(1,1) model with the range-based variances and covariance, included as additional predetermined variables to supplement the lagged squared returns and return cross-products. The results, which we report in column B of Table 4, are striking. All three α estimates are now negligible, and two of three are statistically insignificant. The β estimates, moreover, are smaller, to make room for the δ estimates, which are larger than the α coefficients from model A and highly statistically significant. The upshot is clear: the inclusion of the lagged range-based variances and covariance drives out the lagged squared returns and return cross products, because the former are much more informative about the lagged volatilities than the later.

To understand the role of the range-based estimates in the GARCH framework, and the reason why their inclusion drives out the lagged squared returns and return cross-product, interpret equation (21) as a standard AR process for the variances and covariance. The innovations to the variances and covariance at date *t* are proxied for by the lagged squared returns and return cross-products, with the intuition that when these proxies are large (small), the variances and covariance are likely to have experienced a positive (negative) innovation. Unfortunately, as we explained above, the squared return is a very noisy measure of the variance, suggesting that a substantial part of the volatility variation generated by the GARCH model is driven by proxy noise as opposed to true information about volatility. The range-based estimates, in contrast, are less noisy and therefore serve as better proxies for the variances and covariance innovations.

Finally, in light of the results for model B, we report in column C of Table 4 estimates of a bivariate diagonal GARCH model in which we drop the lagged squared returns and return cross products, and instead include *only* lagged range-based variances and covariances. That is, we impose the restriction $\alpha_1 = \alpha_2 = \alpha_3 = 0$ on model B. The maximized values of the log likelihoods indicate that the restriction can not be rejected at the one percent level; hence, once lagged range-based estimates are included, there appears to be little need to retain lagged squared returns and cross products.[10] In sharp contrast, the likelihood values indicate that the standard GARCH model A is overwhelmingly rejected in favor of a

[10] The p-value is 0.03, suggesting some ambiguity. Note that one would not necessarily expect the model including *only* range-based estimates completely to dominate the model including both range-based estimates and the traditional squared returns and cross products: although the daily range certainly contains information not in the daily return, the daily return also contains information that may not be in the range (the opening and closing prices).

model that includes range-based estimates, whether with or without the traditional squared returns and cross product (models B and C, respectively). In Figure 2, we plot the much smoother estimated volatilities and correlation from our hybrid "range-based GARCH model (C)," which arguably *does* deliver the best of both worlds.

5. Summary and Concluding Remarks

We have extended the important idea of range-based volatility estimation to the multivariate case. In particular, we proposed a range-based covariance estimator motivated by financial economic considerations (the absence of arbitrage), in addition to statistical considerations. We showed that, unlike other univariate and multivariate volatility estimators, the range-based estimator is highly efficient yet robust to market microstructure noise arising from bid-ask bounce and asynchronous trading. Finally, we provided an empirical example illustrating the value of the high-frequency sample path information contained in the range-based estimates in a multivariate GARCH framework.

Many extensions and applications of the ideas developed here are possible; see, for example, Brunetti and Lildolt (2002), who take up several. One intriguing application, which to the best of our knowledge has not yet been explored, involves constructing range-based volatility and covariance bets via portfolios of lookback options. The payoff of a lookback straddle (a lookback call plus a lookback put) is equal to the range of the underlying asset over the life of the option. Therefore, lookback straddles are ideal for placing bets on the volatility of an asset, their payoffs are high (low) when volatility was high (low). Our no-arbitrage approach to covariance estimation suggests an analogous way of placing bets on the *covariance* between two assets. For example, consider a portfolio of a long mark/dollar lookback straddle, a long yen/dollar lookback straddle, and short mark/yen lookback straddle. Given equation (9) and the fact that each of the straddles is a variance bet, the payoffs of this portfolio are high (low) when covariance between the two dollar rates is high (low) over the life of the option.

References

Alizadeh, Sassan, Michael W. Brandt, and Francis X. Diebold, 2002, Range-based estimation of stochastic volatility models, *Journal of Finance* 57, forthcoming.

Andersen, Torben G., and Tim Bollerslev, 1998, Answering the skeptics: Yes, standard volatility models do provide accurate forecasts, *International Economic Review* 39, 885-905.

Andersen, Torben G., Tim Bollerslev, and Francis X. Diebold, 2002, Parametric and nonparametric volatility measurement, in L.P. Hansen and Y. Ait-Sahalia (eds.), *Handbook of Financial Econometrics*. Amsterdam: North-Holland, forthcoming.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Heiko Ebens, 2001, The distribution of realized stock return volatility, *Journal of Financial Economics* 61, 43-76.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Paul Labys, 2001a, The distribution of realized exchange rate volatility, *Journal of the American Statistical Association* 96, 42-55.

Andersen, Torben G., Tim Bollerslev, Francis X. Diebold, and Paul Labys, 2001b, Modeling and forecasting realized volatility, Wharton Financial Institutions Center Working Paper 2001-01 and NBER Working Paper 8160.

Bai, Xuezheng, Jeffrey R. Russell, and George C. Tiao, 2000, Beyond Merton's utopia: Effects of non-normality and dependence on the precision of variance estimates using high-frequency financial data, Manuscript, University of Chicago, Graduate School of Business.

Ball, Clifford A., and Walter N. Torous, 1984, The maximum likelihood estimation of security price volatility: Theory, evidence, and application to option pricing, *Journal of Business* 57, 97-112.

Beckers, Stan, 1983, Variance of security price returns based on high, low, and closing prices, *Journal of Business* 56, 97-112.

Brunetti, Celso, and Peter Lildolt, 2002, Return-based and range-based (co)variance estimation, with an application to foreign exchange markets, manuscript.

Edwards, Robert D., and John Magee, 1997, *Technical Analysis of Stock Trends* (Amacom, New York).

French, Kenneth R., G. William Schwert, and Robert F. Stambaugh, 1987, Expected stock returns and volatility, *Journal of Financial Economics* 19, 3-29.

Garman, Mark B., and Michael J. Klass, 1980, On the estimation of price volatility from historical data, *Journal of Business* 53, 67-78.

Gill, Philip E., Walter Murray, and Margaret H. Wright, 1981, *Practical Optimization*. New York: Academic Press.

Hasbrouck, Joel, 1999a, The dynamics of discrete bid and ask quotes, *Journal of Finance* 54, 2109-2142.

Hasbrouck, Joel, 1999b, Security bid/ask dynamics with discreteness and clustering: Simple strategies for modeling and estimation, *Journal of Financial Markets* 2, 1-28.

Hull, John, and Alan White, 1987, The Pricing of options on assets with stochastic volatilities, *Journal of Finance* 42, 381-400.

Kunitomo, Naoto, 1992, Improving the Parkinson methods of estimating security price volatilities, *Journal of Business* 65, 295-302.

Ledoit, Olivier, and Michael W. Wolf, 2001, A well-conditioned estimator for large-dimensional covariance matrices, Working Paper, Anderson School of Business, UCLA.

Merton, Robert C., 1980, On Estimating the expected return on the market: An exploratory investigation, *Journal of Financial Economics* 8, 323-361.

Parkinson, Michael, 1980, The extreme value method for estimating the variance of the rate of return, *Journal of Business* 53, 61-65.

Rogers, L. Christopher G., and Stephen E. Satchell, 1991, Estimating variance from high, low, and closing prices, *Annals of Applied Probability* 1, 504-512.

Schnabel, Robert B., and Elizabeth Eskow, 1999, A revised modified Cholesky factorization algorithm, *SIAM Journal of Optimization* 9, 1135-1148.

Yang, Dennis, and Qiang Zhang, 2000, Drift-independent volatility estimation based on high, low, open, and close prices, *Journal of Business* 73, 477-491.

Zumbach, G., Corsi, F. and Trapletti, A. (2002), "Efficient Estimation of Volatility using High Frequency Data," Manuscript, Olsen and Associates, Zurich.

Table 1

Range-Based and Realized Estimates in Merton's Utopia

Two dollar denominated exchange rates P_1 and P_2 evolve as drift-less diffusions with annualized volatility σ of 15 percent, covariance of 0.9, and correlation ρ of 0.4. At each instant the cross-currency P_3 is given by the absence of triangular arbitrage as the ratio of the two base currencies. Starting at $P_{1,0}=P_{2,0}=1$, we simulate 24 hours worth of m regularly spaced intraday log prices using $p_{i,t+k/m}=p_{i,t+(k-1)/m}+\sigma\sqrt{250/m}\,\varepsilon_{i,t+k/m}$, $i=\{1,2\}$, and $p_{3,t}=p_{1,t}-p_{2,t}$, for $k=1,...m$, where $p_i=\ln P_i$ and $[\varepsilon_1,\varepsilon_2]$ are standard-normal innovations with correlation ρ. The sampling frequency m ranges from four (one observation every six hours) to 1440 (on observation every minute).

We use this simulated data to compute the daily range and intraday returns and then construct three estimates of the volatilities, covariance, and correlation. We construct range-based covariance estimates using Parkinson's variance estimator and $\text{Cov}[\Delta p_1,\Delta p_2]=1/2(\text{Var}[\Delta p_1]+\text{Var}[\Delta p_2]-\text{Var}[\Delta p_3])$. We construct realized covariance estimates using either the realized variance estimator and the same expression for the covariance or using the cross-products of intraday returns. We repeat this procedure 10,000 times and report the means, standard deviations, and root mean squared errors.

Sampling Frequency	Standard Deviation			Covariance			Correlation		
	Mean	StdDev	RMSE	Mean	StdDev	RMSE	Mean	StdDev	RMSE
	Range-Based Estimates								
1-min	14.099	4.279	4.373	0.862	1.084	1.085	0.371	0.341	0.342
2.5-min	13.891	4.278	4.420	0.839	1.071	1.072	0.370	0.347	0.348
5-min	13.746	4.277	4.457	0.823	1.061	1.064	0.369	0.351	0.352
10-min	13.477	4.274	4.537	0.794	1.043	1.048	0.368	0.359	0.360
20-min	13.090	4.266	4.674	0.753	1.016	1.026	0.366	0.370	0.372
40-min	12.525	4.255	4.923	0.695	0.977	0.998	0.363	0.389	0.391
1hr 20-min	11.701	4.236	5.369	0.615	0.918	0.961	0.358	0.420	0.422
3-hr	10.212	4.177	6.354	0.484	0.808	0.909	0.350	0.485	0.487
6-hr	8.207	4.089	7.929	0.335	0.659	0.868	0.330	0.607	0.611
	Realized Estimates with No-Arbitrage Condition								
1-min	14.997	0.280	0.280	0.900	0.064	0.064	0.400	0.022	0.022
2.5-min	14.992	0.482	0.482	0.900	0.111	0.111	0.400	0.038	0.038
5-min	14.985	0.623	0.624	0.900	0.143	0.143	0.400	0.050	0.050
10-min	14.971	0.883	0.883	0.901	0.202	0.202	0.399	0.070	0.070
20-min	14.943	1.249	1.250	0.900	0.285	0.285	0.398	0.099	0.099
40-min	14.888	1.758	1.762	0.898	0.404	0.404	0.395	0.142	0.142
1hr 20-min	14.788	2.475	2.484	0.896	0.570	0.570	0.389	0.203	0.203
3-hr	14.531	3.684	3.713	0.895	0.858	0.858	0.378	0.311	0.312
6-hr	14.090	5.100	5.181	0.894	1.206	1.206	0.358	0.452	0.454
	Realized Estimates with Cross-Products								
1-min	14.997	0.280	0.280	0.900	0.064	0.064	0.400	0.022	0.022
2.5-min	14.992	0.482	0.482	0.900	0.111	0.111	0.400	0.038	0.038
5-min	14.985	0.623	0.624	0.900	0.143	0.143	0.400	0.050	0.050
10-min	14.971	0.883	0.883	0.901	0.202	0.202	0.399	0.070	0.070
20-min	14.943	1.249	1.250	0.900	0.285	0.285	0.398	0.099	0.099
40-min	14.888	1.758	1.762	0.898	0.404	0.404	0.395	0.142	0.142
1hr 20-min	14.788	2.475	2.484	0.896	0.570	0.570	0.389	0.203	0.203
3-hr	14.531	3.684	3.713	0.895	0.858	0.858	0.378	0.311	0.312
6-hr	14.090	5.100	5.181	0.894	1.206	1.206	0.358	0.452	0.454

Table 2

Range-Based and Realized Estimates with Bid-Ask Bounce

Two dollar denominated exchange rates P_1 and P_2 evolve as drift-less diffusions with annualized volatility σ of 15 percent, covariance of 0.9, and correlation ρ of 0.4. At each instant the cross-currency P_3 is given by the absence of triangular arbitrage as the ratio of the two base currencies. Starting at $P_{1,0}=P_{2,0}=1$, we simulate 24 hours worth of m regularly spaced intraday log prices using $p_{i,t+k/m}=p_{i,t+(k-1)/m}+\sigma\sqrt{250/m}\,\varepsilon_{i,t+k/m}$, $i=\{1,2\}$, for $k=1,...m$, where $p_i=\ln P_i$ and $[\varepsilon_1,\varepsilon_2]$ are standard-normal innovations with correlation ρ. The sampling frequency m ranges from four (one observation every six hours) to 1440 (on observation every minute).

We use these prices to compute the bid and ask quotes $B_{i,t}=\text{floor}[P_{i,t}-1/2\,\text{spread},\text{tick}]$ and $A_{i,t}=\text{ceiling}[P_{i,t}+1/2\,\text{spread},\text{tick}]$, where $\text{floor}[x,\text{tick}]$ and $\text{ceiling}[x,\text{tick}]$ are functions that round x down or up to the nearest tick, respectively. The spread is set to 0.0005 and the tick size is 0.0001. For the cross-currency, we compute the bid and ask quotes by imposing no-arbitrage given the bid and ask quotes of the base currencies. We then take the observed prices as $P_{i,t}^{\text{obs}}=q_{i,t}B_{i,t}+(1-q_{i,t})A_{i,t}$, where $q_{i,t}\sim\text{Bernoulli}[1/2]$. The buy-sell indicators $q_{1,t}$ and $q_{2,t}$ are correlated with $\text{Corr}[q_{1,t},q_{2,t}]=\eta$ but the indicator $q_{3,t}$ is independent. In panel A $\eta=0$, in panel B $\eta=0.5$ and in panel C $\eta=0.75$.

We use this observed data to compute the daily range and intraday returns and then construct three estimates of the volatilities, covariance, and correlation. We construct range-based covariance estimates using Parkinson's variance estimator and $\text{Cov}[\Delta p_1,\Delta p_2]=1/2(\text{Var}[\Delta p_1]+\text{Var}[\Delta p_2]-\text{Var}[\Delta p_3])$. We construct realized covariance estimates using either the realized variance estimator and the same expression for the covariance or using the cross-products of intraday returns. We repeat this procedure 10,000 times and report the means, standard deviations, and root mean squared errors.

Panel A: Independent Bid-Ask Bounce with $\eta = 0$

Sampling Frequency	Standard Deviation			Covariance			Correlation		
	Mean	StdDev	RMSE	Mean	StdDev	RMSE	Mean	StdDev	RMSE
				Range-Based Estimates					
1-min	14.512	4.278	4.306	0.826	1.121	1.124	0.327	0.344	0.352
2.5-min	14.203	4.277	4.351	0.796	1.100	1.105	0.327	0.352	0.360
5-min	14.006	4.274	4.388	0.779	1.087	1.093	0.327	0.357	0.365
10-min	13.671	4.272	4.474	0.754	1.063	1.073	0.331	0.366	0.373
20-min	13.228	4.263	4.617	0.721	1.032	1.047	0.335	0.378	0.384
40-min	12.622	4.256	4.875	0.672	0.989	1.015	0.339	0.397	0.402
1hr 20-min	11.767	4.236	5.328	0.600	0.928	0.975	0.340	0.428	0.432
3-hr	10.249	4.181	6.329	0.476	0.815	0.919	0.336	0.492	0.496
6-hr	8.228	4.093	7.912	0.337	0.675	0.879	0.332	0.606	0.609
				Realized Estimates with No-Arbitrage Condition					
1-min	29.645	0.490	14.653	-5.578	0.462	6.495	-0.636	0.060	1.037
2.5-min	21.036	0.641	6.070	-1.261	0.356	2.190	-0.287	0.087	0.693
5-min	18.849	0.760	3.924	-0.395	0.341	1.339	-0.114	0.100	0.524
10-min	17.010	0.990	2.241	0.253	0.351	0.736	0.082	0.118	0.339
20-min	15.994	1.327	1.658	0.578	0.396	0.511	0.217	0.141	0.231
40-min	15.422	1.820	1.868	0.736	0.486	0.513	0.296	0.177	0.206
1hr 20-min	15.058	2.515	2.516	0.815	0.629	0.635	0.335	0.232	0.241
3-hr	14.649	3.716	3.732	0.859	0.899	0.900	0.348	0.339	0.343
6-hr	14.147	5.120	5.191	0.876	1.236	1.237	0.336	0.490	0.494
				Realized Estimates with Cross-Products					
1-min	29.645	0.490	14.653	0.900	0.263	0.263	0.102	0.030	0.299
2.5-min	21.036	0.641	6.070	0.901	0.218	0.218	0.203	0.046	0.202
5-min	18.849	0.760	3.924	0.900	0.223	0.223	0.253	0.057	0.158
10-min	17.010	0.990	2.241	0.901	0.256	0.256	0.309	0.076	0.119
20-min	15.994	1.327	1.658	0.901	0.322	0.322	0.347	0.104	0.117
40-min	15.422	1.820	1.868	0.898	0.429	0.429	0.368	0.145	0.149
1hr 20-min	15.058	2.515	2.516	0.896	0.588	0.588	0.376	0.205	0.207
3-hr	14.649	3.716	3.732	0.895	0.870	0.870	0.371	0.312	0.314
6-hr	14.147	5.120	5.191	0.894	1.215	1.215	0.355	0.453	0.455

Panel B: Correlated Bid-Ask Bounce with η=0.5

Sampling Frequency	Standard Deviation			Covariance			Correlation		
	Mean	StdDev	RMSE	Mean	StdDev	RMSE	Mean	StdDev	RMSE
	Range-Based Estimates								
1-min	14.512	4.278	4.306	0.810	1.123	1.127	0.318	0.347	0.356
2.5-min	14.203	4.277	4.351	0.781	1.102	1.108	0.318	0.355	0.364
5-min	14.006	4.274	4.388	0.764	1.089	1.097	0.319	0.360	0.369
10-min	13.671	4.272	4.474	0.741	1.065	1.077	0.323	0.369	0.377
20-min	13.228	4.263	4.617	0.710	1.033	1.051	0.329	0.380	0.387
40-min	12.622	4.256	4.875	0.664	0.990	1.018	0.333	0.399	0.405
1hr 20-min	11.767	4.236	5.328	0.595	0.929	0.978	0.335	0.430	0.435
3-hr	10.249	4.181	6.329	0.473	0.816	0.921	0.332	0.494	0.499
6-hr	8.228	4.093	7.912	0.334	0.679	0.884	0.317	0.601	0.607
	Realized Estimates with No-Arbitrage Condition								
1-min	29.645	0.490	14.653	-7.812	0.548	8.729	-0.890	0.073	1.292
2.5-min	21.036	0.641	6.070	-2.006	0.398	2.933	-0.456	0.102	0.862
5-min	18.849	0.760	3.924	-0.842	0.375	1.782	-0.241	0.114	0.651
10-min	17.010	0.990	2.241	0.029	0.375	0.949	0.004	0.130	0.417
20-min	15.994	1.327	1.658	0.465	0.414	0.601	0.172	0.151	0.273
40-min	15.422	1.820	1.868	0.679	0.500	0.547	0.270	0.185	0.226
1hr 20-min	15.058	2.515	2.516	0.786	0.640	0.650	0.321	0.238	0.251
3-hr	14.649	3.716	3.732	0.846	0.906	0.908	0.341	0.343	0.348
6-hr	14.147	5.120	5.191	0.869	1.241	1.242	0.331	0.493	0.498
	Realized Estimates with Cross-Products								
1-min	29.645	0.490	14.653	4.140	0.265	3.251	0.471	0.024	0.075
2.5-min	21.036	0.641	6.070	1.980	0.224	1.103	0.447	0.039	0.061
5-min	18.849	0.760	3.924	1.549	0.230	0.688	0.435	0.050	0.061
10-min	17.010	0.990	2.241	1.225	0.262	0.418	0.421	0.070	0.073
20-min	15.994	1.327	1.658	1.062	0.328	0.366	0.410	0.099	0.099
40-min	15.422	1.820	1.868	0.979	0.433	0.440	0.401	0.141	0.141
1hr 20-min	15.058	2.515	2.516	0.937	0.591	0.592	0.393	0.202	0.202
3-hr	14.649	3.716	3.732	0.913	0.872	0.872	0.379	0.311	0.311
6-hr	14.147	5.120	5.191	0.904	1.216	1.216	0.358	0.452	0.454

Panel C: Correlated Bid-Ask Bounce with η=0.75

Sampling Frequency	Standard Deviation Mean	StdDev	RMSE	Covariance Mean	StdDev	RMSE	Correlation Mean	StdDev	RMSE
				Range-Based Estimates					
1-min	14.512	4.278	4.306	0.810	1.123	1.127	0.318	0.347	0.356
2.5-min	14.203	4.277	4.351	0.781	1.102	1.109	0.318	0.355	0.364
5-min	14.006	4.274	4.388	0.764	1.089	1.097	0.319	0.360	0.369
10-min	13.671	4.272	4.474	0.741	1.065	1.077	0.323	0.369	0.377
20-min	13.228	4.263	4.617	0.710	1.034	1.051	0.328	0.380	0.387
40-min	12.622	4.256	4.875	0.664	0.991	1.018	0.333	0.399	0.405
1hr 20-min	11.767	4.236	5.328	0.595	0.929	0.978	0.335	0.430	0.435
3-hr	10.249	4.181	6.329	0.473	0.817	0.921	0.332	0.494	0.499
6-hr	8.228	4.093	7.912	0.330	0.662	0.874	0.318	0.603	0.609
				Realized Estimates with No-Arbitrage Condition					
1-min	29.645	0.490	14.653	-7.812	0.555	8.729	-0.890	0.075	1.292
2.5-min	21.036	0.641	6.070	-2.006	0.403	2.934	-0.456	0.104	0.863
5-min	18.849	0.760	3.924	-0.842	0.379	1.783	-0.241	0.116	0.652
10-min	17.010	0.990	2.241	0.028	0.378	0.950	0.003	0.131	0.418
20-min	15.994	1.327	1.658	0.465	0.417	0.603	0.172	0.152	0.274
40-min	15.422	1.820	1.868	0.679	0.502	0.549	0.270	0.186	0.227
1hr 20-min	15.058	2.515	2.516	0.786	0.641	0.651	0.321	0.238	0.251
3-hr	14.649	3.716	3.732	0.846	0.907	0.908	0.340	0.343	0.348
6-hr	14.147	5.120	5.191	0.869	1.242	1.243	0.331	0.491	0.496
				Realized Estimates with Cross-Products					
1-min	29.650	0.490	14.653	5.761	0.263	4.868	0.655	0.019	0.256
2.5-min	21.036	0.641	6.070	2.520	0.225	1.636	0.569	0.033	0.172
5-min	18.849	0.760	3.924	1.873	0.233	1.000	0.526	0.044	0.133
10-min	17.010	0.990	2.241	1.387	0.265	0.554	0.476	0.065	0.100
20-min	15.994	1.327	1.658	1.143	0.330	0.410	0.441	0.095	0.104
40-min	15.422	1.820	1.868	1.019	0.436	0.452	0.418	0.139	0.140
1hr 20-min	15.058	2.515	2.516	0.957	0.593	0.596	0.401	0.201	0.201
3-hr	14.649	3.716	3.732	0.922	0.873	0.873	0.383	0.310	0.310
6-hr	14.147	5.120	5.191	0.908	1.217	1.217	0.360	0.451	0.453

Table 3

Range-Based and Realized Estimates with Asynchronous Trading

Two dollar denominated exchange rates P_1 and P_2 evolve as drift-less diffusions with annualized volatility σ of 15 percent, covariance of 0.9, and correlation ρ of 0.4. At each instant the cross-currency P_3 is given by the absence of triangular arbitrage as the ratio of the two base currencies. Starting at $P_{1,0}=P_{2,0}=P_{3,0}=1$, we simulate 24 hours worth of $m=17280$ regularly spaced intraday log prices (one price every second) using $p_{i,t+k/m}=p_{i,t+(k-1)/m}+\sigma\sqrt{250/m}\,\varepsilon_{i,t+k/m}$, $i=\{1,2\}$, and $p_{3,t}=p_{1,t}-p_{2,t}$, for $k=1,...m$, where $p_i=\ln P_i$ and $[\varepsilon_1,\varepsilon_2]$ are standard-normal innovations with correlation ρ.

We assign to each process *n* trades times randomly throughout the day and construct stale price processes for which the price is equal to the price at the previous trade time until it is reset to the latent true price at the next trade time. In panel A $n=288$ (an average of one trade every five minutes) and in panel B $n=1440$ (an average of one trade every minute). We then sample these stale price processes at a regular frequency *m* ranging from four (one observation every six hours) to 1440 (one observation every minute).

We use this regularly sampled data to compute the daily range and intraday returns and then construct three estimates of the volatilities, covariance, and correlation. We construct range-based covariance estimates using Parkinson's variance estimator and $\mathrm{Cov}[\Delta p_1,\Delta p_2]=1/2(\mathrm{Var}[\Delta p_1]+\mathrm{Var}[\Delta p_2]-\mathrm{Var}[\Delta p_3])$. We construct realized covariance estimates using either the realized variance estimator and the same expression for the covariance or using the cross-products of intraday returns. We repeat this procedure 10,000 times and report the means, standard deviations, and root mean squared errors.

Panel A: 288 Trades per Day

Sampling Frequency	Standard Deviation			Covariance			Correlation		
	Mean	StdDev	RMSE	Mean	StdDev	RMSE	Mean	StdDev	RMSE
	Range-Based Estimates								
1-min	13.629	4.328	4.538	0.846	1.088	1.089	0.380	0.351	0.351
2.5-min	13.608	4.329	4.545	0.844	1.087	1.088	0.379	0.352	0.352
5-min	13.557	4.315	4.548	0.838	1.080	1.082	0.379	0.353	0.353
10-min	13.371	4.305	4.601	0.816	1.072	1.075	0.378	0.361	0.361
20-min	13.029	4.307	4.734	0.786	1.047	1.052	0.380	0.369	0.369
40-min	12.509	4.300	4.967	0.731	1.009	1.022	0.379	0.391	0.392
1hr 20-min	11.651	4.252	5.411	0.652	0.960	0.991	0.375	0.429	0.429
3-hr	10.177	4.069	6.309	0.508	0.834	0.922	0.365	0.494	0.495
6-hr	8.237	3.938	7.826	0.342	0.671	0.873	0.348	0.585	0.587
	Realized Estimates with No-Arbitrage Condition								
1-min	14.959	0.882	0.882	0.899	0.215	0.215	0.400	0.076	0.076
2.5-min	14.959	0.894	0.895	0.898	0.218	0.218	0.400	0.077	0.077
5-min	14.961	0.907	0.907	0.899	0.220	0.220	0.400	0.078	0.077
10-min	14.973	1.077	1.077	0.894	0.261	0.261	0.397	0.092	0.092
20-min	14.929	1.360	1.362	0.883	0.324	0.325	0.392	0.118	0.118
40-min	14.900	1.860	1.862	0.884	0.433	0.433	0.390	0.160	0.161
1hr 20-min	14.824	2.531	2.536	0.910	0.614	0.614	0.391	0.225	0.225
3-hr	14.598	3.577	3.598	0.914	0.919	0.918	0.367	0.368	0.370
6-hr	14.271	5.152	5.201	0.939	1.255	1.255	0.348	0.592	0.594
	Realized Estimates with Cross-Products								
1-min	14.959	0.882	0.882	0.046	0.089	0.859	0.021	0.040	0.382
2.5-min	14.959	0.894	0.895	0.086	0.121	0.823	0.038	0.054	0.366
5-min	14.961	0.907	0.907	0.114	0.149	0.800	0.051	0.066	0.355
10-min	14.973	1.077	1.077	0.195	0.217	0.738	0.087	0.095	0.327
20-min	14.929	1.360	1.362	0.322	0.305	0.653	0.143	0.133	0.289
40-min	14.900	1.860	1.862	0.481	0.422	0.595	0.214	0.182	0.260
1hr 20-min	14.824	2.531	2.536	0.654	0.578	0.628	0.284	0.229	0.257
3-hr	14.598	3.577	3.598	0.821	0.848	0.851	0.339	0.320	0.326
6-hr	14.271	5.152	5.201	0.892	1.188	1.188	0.350	0.459	0.462

Panel B: 1440 Trades per Day

Sampling Frequency	Standard Deviation			Covariance			Correlation		
	Mean	StdDev	RMSE	Mean	StdDev	RMSE	Mean	StdDev	RMSE
	Range-Based Estimates								
1-min	14.037	4.333	4.436	0.894	1.115	1.115	0.382	0.341	0.341
2.5-min	13.872	4.333	4.475	0.872	1.107	1.106	0.380	0.347	0.348
5-min	13.743	4.335	4.511	0.858	1.098	1.098	0.379	0.350	0.351
10-min	13.472	4.315	4.575	0.826	1.078	1.079	0.377	0.359	0.359
20-min	13.096	4.308	4.708	0.789	1.050	1.055	0.377	0.367	0.368
40-min	12.531	4.280	4.939	0.730	1.017	1.030	0.374	0.391	0.392
1hr 20-min	11.674	4.250	5.395	0.646	0.958	0.991	0.368	0.428	0.429
3-hr	10.212	4.122	6.316	0.516	0.837	0.920	0.373	0.485	0.486
6-hr	8.279	3.997	7.819	0.349	0.667	0.865	0.345	0.592	0.595
	Realized Estimates with No-Arbitrage Condition								
1-min	14.989	0.405	0.405	0.898	0.097	0.097	0.399	0.034	0.034
2.5-min	14.964	0.559	0.560	0.891	0.133	0.133	0.397	0.046	0.046
5-min	14.952	0.678	0.680	0.888	0.163	0.163	0.395	0.057	0.057
10-min	14.940	0.932	0.934	0.891	0.222	0.222	0.396	0.079	0.079
20-min	14.923	1.292	1.294	0.882	0.315	0.315	0.390	0.113	0.113
40-min	14.898	1.773	1.775	0.898	0.433	0.433	0.394	0.158	0.158
1hr 20-min	14.836	2.463	2.468	0.913	0.591	0.591	0.391	0.220	0.220
3-hr	14.680	3.597	3.609	0.924	0.867	0.867	0.379	0.328	0.329
6-hr	14.255	5.138	5.189	0.936	1.214	1.214	0.361	0.483	0.484
	Realized Estimates with Cross-Products								
1-min	14.989	0.405	0.405	0.096	0.096	0.810	0.043	0.042	0.360
2.5-min	14.964	0.559	0.560	0.205	0.172	0.716	0.091	0.076	0.318
5-min	14.952	0.678	0.680	0.266	0.212	0.668	0.119	0.094	0.296
10-min	14.940	0.932	0.934	0.387	0.257	0.574	0.173	0.112	0.253
20-min	14.923	1.292	1.294	0.545	0.317	0.476	0.243	0.133	0.206
40-min	14.898	1.773	1.775	0.700	0.402	0.449	0.310	0.161	0.185
1hr 20-min	14.836	2.463	2.468	0.824	0.558	0.563	0.356	0.210	0.214
3-hr	14.680	3.597	3.609	0.881	0.851	0.851	0.364	0.318	0.319
6-hr	14.255	5.138	5.189	0.934	1.199	1.199	0.364	0.459	0.460

Table 4

Multivariate GARCH(1,1) Estimates

We present maximum likelihood estimates of the bivariate diagonal GARCH(1,1) model, potentially with range-based variance/covariance variables included in the variance/covariance equations. The model is:

$$\begin{bmatrix} \Delta p_{1,t+1} \\ \Delta p_{2,t+1} \end{bmatrix} = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix} + \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix} \quad \text{with} \quad \mathrm{Var}_t \begin{bmatrix} \varepsilon_{1,t+1} \\ \varepsilon_{2,t+1} \end{bmatrix} = \begin{bmatrix} h_{1,t}^2 & h_{12,t} \\ h_{12,t} & h_{2,t}^2 \end{bmatrix}$$

and the volatility dynamics:

$$h_{1,t}^2 = \omega_1 + \beta_1 h_{1,t-1}^2 + \alpha_1 \varepsilon_{1,t-1}^2 + \delta_1 x_{1,t-1}^2$$

$$h_{2,t}^2 = \omega_2 + \beta_2 h_{2,t-1}^2 + \alpha_2 \varepsilon_{2,t-1}^2 + \delta_2 x_{2,t-1}^2$$

$$h_{12,t} = \omega_3 + \beta_3 h_{12,t-1} + \alpha_3 \varepsilon_{1,t-1} \varepsilon_{2,t-1} + \delta_3 x_{12,t-1},$$

where Δp_1 is the mark/dollar return, Δp_2 is the yen/dollar return, x_1, x_2, and x_{12} are the range-based mark and yen return volatilities and covariance, and the return innovations are conditionally Gaussian.

The data are daily returns and range-based volatility estimates for the front-month mark and yen and futures contracts. The cross-rate ranges are constructed from future transactions data as described in the text. The sample period is January 3, 1990 through December 30, 1998 (2,189 observations).

	Observable Predetermined Variables Included in Variance / Covariance Equations					
	(A) Lagged Return Square or Cross Product		(B) Lagged Return Square or Cross Product, and Range-Based Var. or Cov.		(C) Range-Based Var. or Cov.	
Parameter	Estimate	*t*-Stat	Estimate	*t*-Stat	Estimate	*t*-Stat
μ_1	$-3.63\text{x}10^{-5}$	-0.311	$-9.75\text{x}10^{-5}$	-0.840	$-9.83\text{x}10^{-5}$	-0.852
μ_2	$-7.59\text{x}10^{-6}$	-0.059	$-4.80\text{x}10^{-5}$	-0.392	$-5.01\text{x}10^{-5}$	-0.373
ω_1	$6.45\text{x}10^{-7}$	5.563	$6.40\text{x}10^{-7}$	4.850	$6.59\text{x}10^{-7}$	4.793
ω_2	$1.03\text{x}10^{-6}$	8.229	$1.21\text{x}10^{-6}$	7.210	$1.31\text{x}10^{-6}$	6.426
ω_3	$4.15\text{x}10^{-7}$	5.232	$4.89\text{x}10^{-7}$	4.820	$5.03\text{x}10^{-7}$	5.440
β_1	0.942	163.931	0.920	111.970	0.921	110.492
β_2	0.931	192.835	0.889	121.340	0.887	119.963
β_3	0.940	222.936	0.916	145.010	0.918	142.214
α_1	0.052	10.937	-0.008	-1.217		
α_2	0.055	19.873	0.013	1.930		
α_3	0.051	14.830	0.008	1.350		
δ_1			0.099	7.493	0.096	11.369
δ_2			0.084	6.661	0.110	15.247
δ_3			0.085	8.824	0.099	13.331
Log Likelihood	21431.45		21474.92		21470.31	

Figure 1

Range-Based Volatilities and Correlation

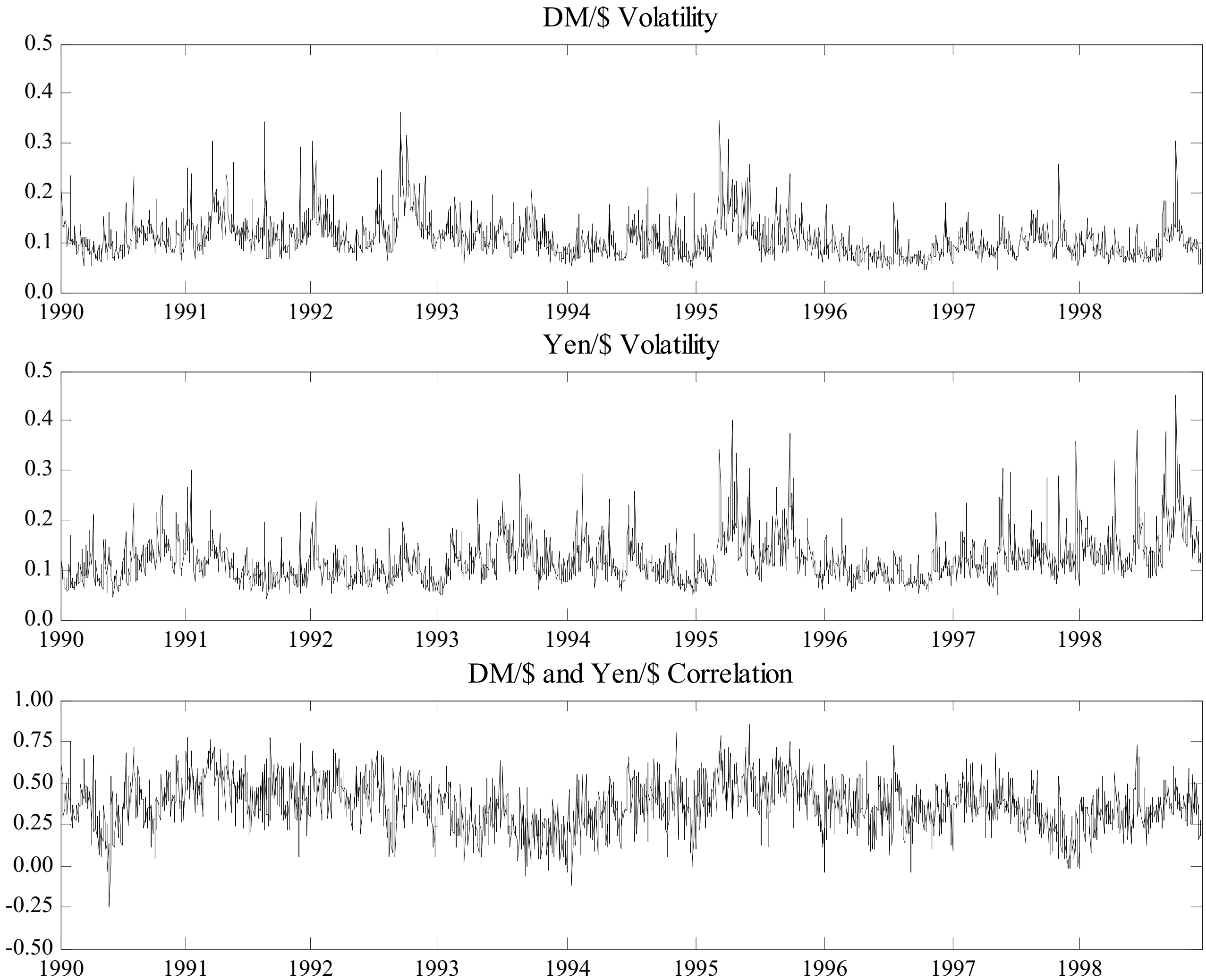
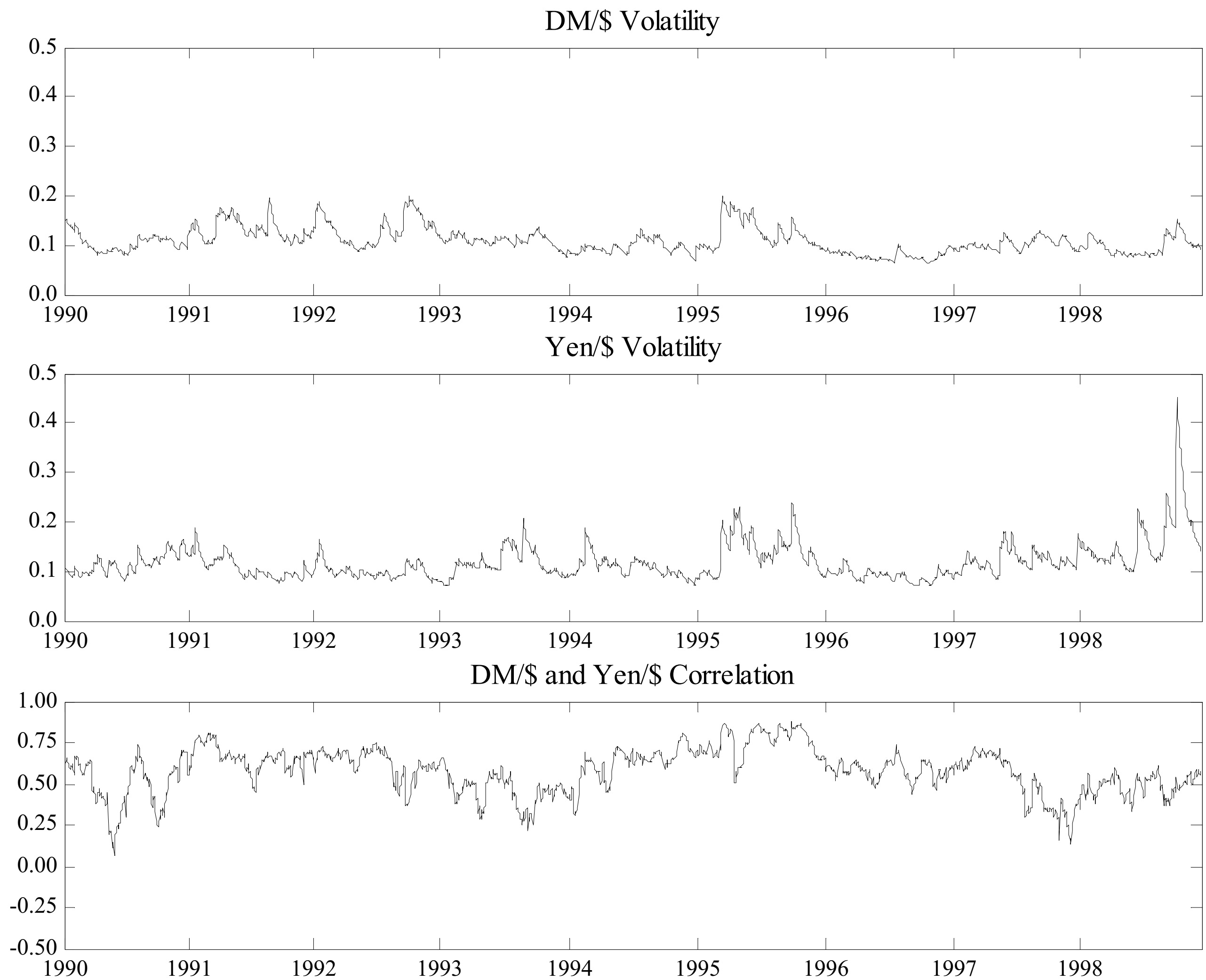


Figure 2

Bivariate GARCH(0,1) Volatilities and Correlation with Range-Based Estimates as Exogenous Predictors



CFS Working Paper Series:

No.	Author(s)	Title
2003/45	**Guenter W. Beck**	Nominal Exchange Rate Regimes and Relative Price Dispersion: On the Importance of Nominal Exchange Rate Volatility for the Width of the Border
2003/46	**Michael Ehrmann** **Marcel Fratzscher**	Equal size, equal role? Interest rate interdependence between the euro area and the United States
2003/47	**Romain Bouis**	IPOs Cycle and Investment in High-Tech Industries
2003/48	**Martin D. Dietz**	Screening and Advising by a Venture Capitalist with a Time Constraint
2004/01	**Ivica Dus** **Raimond Maurer** **Olivia S. Mitchell**	Betting on Death and Capital Markets in Retirement: A Shortfall Risk Analysis of Life Annuities versus Phased Withdrawal Plans
2004/02	**Tereza Tykvová** **Uwe Walz**	Are IPOs of Different VCs Different?
2004/03	**Marc Escrihuela-Villar**	Innovation and Market Concentration with Asymmetric Firms
2004/04	**Ester Faia** **Tommaso Monacelli**	Ramsey Monetary Policy and International Relative Prices
2004/05	**Uwe Wals** **Douglas Cumming**	Private Equity Returns and Disclosure around the World
2004/06	**Dorothea Schäfer** **Axel Werwatz** **Volker Zimmermann**	The Determinants of Debt and (Private-) Equity Financing in Young Innovative SMEs: Evidence from Germany
2004/07	**Michael W. Brandt** **Francis X. Diebold**	A No-Arbitrage Approach to Range-Based Estimation of Return Covariances and Correlations

Copies of working papers can be downloaded at http://www.ifk-cfs.de