Prediction in Financial Markets

• Theories
  – Usually require assumptions about the probability distribution of future returns
  – Efficient markets suggests predicting future outcomes is very difficult, if not impossible
  – There are anomalies (a few basis points here and there), which behaviorists seize on to argue for their perspective

• Practice
  – A lot of demand for predictions
  – A lot of supply of predictions

• Consider different markets
  – Fixed Income
  – Equity
**Forecasting in Bond Markets**

- Interest rates derived from bond prices are used to predict future interest rates
  - Forward rates are the differences between the interest rates on bonds of different maturities
  - E.g., the difference between the 5-year yield and the 4-year yield is a measure of what the ‘market’ ‘thinks’ one year interest rates will be 4 years from now
- Forward rates are biased predictors of future interest rates
  - They embed a risk premium for undertaking long term obligations
  - Finding the risk premium is addressed by estimating the parameters of an econometric interest rate model and making ad hoc assumptions about how interest rates evolve over time
- This only gets a ‘point’ forecast, not the whole distribution that governs the uncertainty of future interest rates

**Forecasting in Equities Markets**

- Expected returns
  - Simple statistics
    - Use historical market returns and the historical premium over risk-free returns to predict stock returns
  - Statistical models
    - Use econometric models, e.g., distributed lag models or a dividend/yield model to predict stock returns
  - Price/Earnings (P/E) model
    - Assumes that when the P/E ratio is high or low relative to some historical norm, then prices will fall or rise so that the ratio will revert back to the norm
- Volatility
  - Use historical standard deviations or the current VIX
- My favorite: Surveys of market participants and institutional peers
An Alternative Approach to Forecasting

• An amazing variety of securities are traded in rich financial markets
  – Some have maturities that extend far into the future
  – Some have terms that are contingent on future events

• Surely the prices of these instruments contain information about the
distribution of future returns

• Let’s look closely at the derivatives and options markets and the
models we currently use to understand them

The Options Market

• **Put** options pay off when the market price of a given stock (underlying) is
  *below* a given amount (strike) at a given time in the future (maturity)
• **Call** options pay off when the market price of a given stock (underlying) is
  *above* a given amount (strike) at a given time in the future (maturity)
  – E.g., on September 15, 2015 Netflix stock closed at $104.15, and options with
    a maturity date of November 20, 2015 and with a strike of $100 had these
    prices: Call $12.82 Put $8.55.

• More important than options on one stock, options whose underlying
  instruments are broad market indexes are traded
  – E.g., on September 15, 2015 the SPY, an S&P 500 ETF traded at $199.73, and
    options with a maturity date of November 20, 2015 and with a strike of $200
    had these prices: Call $5.60 Put $6.16.
The Volatility Surface

The Binomial Model

- Arguably, the Black-Merton-Scholes (‘BMS’) model and its close cousin, the binomial model, are the most successful models in economics if not in all of the social sciences:

\[
S \quad \frac{f}{1-f} \quad aS \quad bS
\]

\[ b < 1 + r < a, \]
\[ r \text{ is the risk free interest rate} \]

- The absence of arbitrage – a free lunch - implies that there are positive prices for pure securities that pay 1 euro if the stock goes up or 1 euro if it goes down:

\[
p(a) = \left( \frac{1}{1+r} \right) \frac{(1+r)-b}{a-b} \quad \text{and} \quad p(b) = \left( \frac{1}{1+r} \right) \frac{a-(1+r)}{a-b}
\]
The Binomial Model

- Since these are the prices of securities that pay 1 euro if the stock goes up or down, the price, $D$, of any derivative security (e.g., a put or a call) with payoffs of $D(a)$ if the stock goes up and $D(b)$ if it goes down is simply

$$D = p(a)D(a) + p(b)D(b) = \left(\frac{1}{1 + r}\right)[\pi D(a) + (1 - \pi)D(b)]$$

$$\pi = \frac{(1+r)-b}{a-b} = 'risk\ neutral'\ or\ 'martingale'\ probability\ of\ an\ up\ move$$

- Strikingly, the natural probability of going up, $f$, plays no role whatsoever in this theory for pricing derivatives

- Conversely, there is no way to use derivatives prices to find any information about the true but unknown probability that the stock goes up, $f$

Black-Merton-Scholes*

(*an MIT page)

- The binomial model converges to a lognormal diffusion as the step size gets smaller

$$\frac{dS}{S} = \mu dt + \sigma dz$$

- The binomial formula for the price of an option also converges to the famous BMS risk neutral differential equation

$$\frac{1}{2}\sigma^2S^2 \frac{\partial^2 P}{\partial S^2} + rS \frac{\partial P}{\partial S} - rP + \frac{\partial P}{\partial t} = 0$$

yielding the equally famous BMS solution for a call option

$$P(S,t) = SN(d_1) - e^{-r(t-t)}N(d_2)$$

where $N(\cdot)$ is the cumulative normal, $K$ is the strike, and

$$d_1 = \frac{\ln\left(\frac{S}{K}\right) + (r - \frac{1}{2})\sigma^2(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}$$
Since the expected return on the stock, $\mu$, doesn’t enter the above formula for the price of a call, *option prices are independent of the expected return* of the stock in the BMS model.

This is equal parts:

- Blessing: We can price derivatives without knowing the very difficult to measure expected return;
- Curse: It renders the market for derivatives useless for determining the natural distribution

The challenge, then, is to see if it is possible to *build a different class of models* with the ability to *link option prices to the underlying parameters of the stock process* while also *being consistent with everything else we’ve learned about options*.

Puts are insurance and like any insurance, their prices are a product of three effects:

"Put price = Discounting x Risk Aversion x Probability of a Crash"

If we know the discount rate that leaves risk aversion and probability to determine the put price

But which is it – if the price is high, then how much of the high price comes from high risk aversion and how much from a high probability of a crash?

The binomial model and BMS not only provide no answer to this question, *they also suggest that no answer is possible*. Because of the success of option pricing theories, to some extent we have lost the basic intuitions of insurance in derivatives markets.
A Different Model: Recovery Theorem

- A state is a description of all the information the market uses to forecast the market
  - e.g., the current market level, last month’s returns, current implied volatility
- \( f(i,j) \), is the probability that the economy moves from state \( i \) to state \( j \) in, say, the coming quarter
- \( p(i,j) \), is the contingent price of a security that pays 1 euro if the system is currently in state \( i \) and transits to state \( j \)
- We can find the \( p \)'s from put and call prices, and we want to use the \( p \)'s to find the \( f \)'s
- The next slide describes the relation between these prices, \( p(i,j) \), and the natural probabilities, \( f(i,j) \) we are trying to estimate

Risk Aversion, Discounting and Contingent Prices

- Since a contingent security pays if the world goes from state \( i \) to state \( j \), it protects or insures against the consequences of state \( j \)
- And, like any insurance, contingent prices are the product:
  \[
p(i,j) = \delta \phi(i,j) f(i,j)
\]
  where \( \delta \) is the discount factor, \( \phi \) is the pricing ‘kernel’ that captures risk aversion and the risk premium, and \( f(i,j) \) is the natural transition probability
- The following slide shows how to use the contingent prices, \( p(i,j) \), to find the discount factor, \( \delta \), risk aversion, \( \phi \), and the natural probabilities, \( f(i,j) \)
The Recovery Theorem

• The pricing equation above has the form:

\[ p(i,j) = \delta \phi(i,j) f(i,j) \]

• Only knowing \( p \), there is no way in general to disentangle the kernel from the probabilities
  – There are \( 2m^2 + 1 \) unknowns and only \( m^2 + m \) equations

• Suppose, though, that the kernel has the form:

\[ \phi(i,j) = \phi(j)/\phi(i) \]

• Now we have only \( m^2 + m + 1 \) unknowns and \( m^2 + m \) equations and we can solve for \( \delta, \phi \) and \( f \)

Applying the Recovery Theorem:
A Three Step Procedure

First, pick a date. Then proceed as follows

• Step 1: Use option prices to get pure securities prices

• Step 2: Use the pure prices to find the contingent prices

• Step 3: Apply the Recovery Theorem to determine the risk aversion – the pricing kernel – and the market’s probabilities for equity returns as of that date
Step 1: Pure Security Prices for $1 Contingent Payoffs

<table>
<thead>
<tr>
<th>Tenor</th>
<th>0.25</th>
<th>0.50</th>
<th>0.75</th>
<th>1.00</th>
<th>1.25</th>
<th>1.50</th>
<th>1.75</th>
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<td>$0.005</td>
<td>$0.023</td>
<td>$0.038</td>
<td>$0.050</td>
<td>$0.058</td>
<td>$0.064</td>
<td>$0.068</td>
<td>$0.071</td>
<td>$0.073</td>
<td>$0.076</td>
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<tr>
<td>-29%</td>
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<td>$0.019</td>
<td>$0.026</td>
<td>$0.030</td>
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<td>$0.034</td>
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<tr>
<td>-23%</td>
<td>$0.018</td>
<td>$0.041</td>
<td>$0.046</td>
<td>$0.050</td>
<td>$0.051</td>
<td>$0.052</td>
<td>$0.052</td>
<td>$0.051</td>
<td>$0.050</td>
<td>$0.049</td>
<td>$0.048</td>
<td>$0.046</td>
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<td>Market</td>
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<tr>
<td>-16%</td>
<td>$0.045</td>
<td>$0.064</td>
<td>$0.073</td>
<td>$0.073</td>
<td>$0.072</td>
<td>$0.070</td>
<td>$0.068</td>
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<td>$0.058</td>
<td>$0.056</td>
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<td>Scenario</td>
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<td>$0.156</td>
<td>$0.142</td>
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<td>$0.118</td>
<td>$0.109</td>
<td>$0.102</td>
<td>$0.096</td>
<td>$0.091</td>
<td>$0.085</td>
<td>$0.081</td>
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<tr>
<td></td>
<td>0%</td>
<td>$0.478</td>
<td>$0.302</td>
<td>$0.234</td>
<td>$0.198</td>
<td>$0.173</td>
<td>$0.155</td>
<td>$0.141</td>
<td>$0.129</td>
<td>$0.120</td>
<td>$0.111</td>
<td>$0.103</td>
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<td></td>
<td>9%</td>
<td>$0.276</td>
<td>$0.316</td>
<td>$0.278</td>
<td>$0.245</td>
<td>$0.219</td>
<td>$0.198</td>
<td>$0.180</td>
<td>$0.164</td>
<td>$0.151</td>
<td>$0.140</td>
<td>$0.130</td>
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<tr>
<td></td>
<td>19%</td>
<td>$0.007</td>
<td>$0.070</td>
<td>$0.129</td>
<td>$0.155</td>
<td>$0.166</td>
<td>$0.167</td>
<td>$0.164</td>
<td>$0.158</td>
<td>$0.152</td>
<td>$0.145</td>
<td>$0.137</td>
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<td></td>
<td>30%</td>
<td>$0.000</td>
<td>$0.002</td>
<td>$0.016</td>
<td>$0.036</td>
<td>$0.055</td>
<td>$0.072</td>
<td>$0.085</td>
<td>$0.094</td>
<td>$0.100</td>
<td>$0.103</td>
<td>$0.105</td>
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<tr>
<td></td>
<td>41%</td>
<td>$0.000</td>
<td>$0.000</td>
<td>$0.001</td>
<td>$0.004</td>
<td>$0.009</td>
<td>$0.017</td>
<td>$0.026</td>
<td>$0.036</td>
<td>$0.045</td>
<td>$0.053</td>
<td>$0.061</td>
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<tr>
<td></td>
<td>54%</td>
<td>$0.000</td>
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<td>$0.000</td>
<td>$0.000</td>
<td>$0.000</td>
<td>$0.000</td>
<td>$0.001</td>
<td>$0.001</td>
<td>$0.002</td>
<td>$0.002</td>
<td>$0.003</td>
</tr>
</tbody>
</table>

Priced using the SPX volatility surface from April 27, 2011

- A pure security price is the price of a security that pays one dollar in a given market scenario for a given tenor - for example, a security that pays $1 if the market is unchanged (0% scenario) in 6 months costs $0.302

The Pure Security Price Surface

Priced using the SPX volatility surface from April 27, 2011
Step 2: Estimate Contingent Prices

- The prices we need are the contingent prices, \( p(i,j) \) for moving from state \( i \) to state \( j \) in, say, the next month.

- The next slide illustrates how to get these contingent prices, \( p(i,j) \), from the pure state prices.

- We find the contingent prices by moving through the table of pure security prices along possible market paths.

- The approach is an application of the forward equation for the conservation of probability.
Contingent Forward Prices
Quarterly

Contingent Forward Prices

<table>
<thead>
<tr>
<th>Market Scenario</th>
<th>Final Period</th>
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<tr>
<td></td>
<td>-35%</td>
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<tr>
<td>-35%</td>
<td>$0.671</td>
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<td>-23%</td>
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<td>-16%</td>
<td>$0.006</td>
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<tr>
<td>-8%</td>
<td>$0.006</td>
</tr>
<tr>
<td>0%</td>
<td>$0.005</td>
</tr>
<tr>
<td>9%</td>
<td>$0.001</td>
</tr>
<tr>
<td>19%</td>
<td>$0.001</td>
</tr>
<tr>
<td>30%</td>
<td>$0.002</td>
</tr>
<tr>
<td>41%</td>
<td>$0.001</td>
</tr>
<tr>
<td>54%</td>
<td>$0.000</td>
</tr>
</tbody>
</table>

Priced using the SPX volatility surface from April 27, 2011

• Contingent forward prices are the prices of securities that pay one dollar in a given future market scenario, given the current market range - for example, a security that pays $1 if the market is down 16% next quarter given that the market is currently down 8% costs $0.211

Step 3: Apply the Recovery Theorem to P

• We can now apply the Recovery Theorem and solve for
  – δ, the discount factor
  – ϕ, risk aversion
  and
  – the true probabilities, f(i,j)

• The next slides compare these predictions for a given date with the historical probabilities
Implied Market Utility Function:
The Pricing Kernel on May 1, 2009

Stock Market Return Relative
Recovered Density
May 1, 2009

Recovered Probabilities vs. Risk-Neutral Probabilities on May 1, 2009
### Recovered Probabilities vs. Historical Probabilities on May 1, 2009

![Graph showing recovered and historical probabilities]

### Bootstrapped (Historical) and Recovered Probabilities On May 1, 2009 for May 1, 2010

<table>
<thead>
<tr>
<th>Market Scenario</th>
<th>Historical Bootstrap</th>
<th>Recovered Probabilities</th>
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<tr>
<td>-50%</td>
<td>0.005</td>
<td>0.017</td>
</tr>
<tr>
<td>-40%</td>
<td>0.017</td>
<td>0.041</td>
</tr>
<tr>
<td>-30%</td>
<td>0.045</td>
<td>0.083</td>
</tr>
<tr>
<td>-20%</td>
<td>0.114</td>
<td>0.146</td>
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<tr>
<td>-10%</td>
<td>0.233</td>
<td>0.234</td>
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<tr>
<td>-5%</td>
<td>0.315</td>
<td>0.288</td>
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<tr>
<td>0%</td>
<td>0.407</td>
<td>0.349</td>
</tr>
<tr>
<td>5%</td>
<td>0.507</td>
<td>0.416</td>
</tr>
<tr>
<td>10%</td>
<td>0.605</td>
<td>0.487</td>
</tr>
<tr>
<td>20%</td>
<td>0.762</td>
<td>0.631</td>
</tr>
<tr>
<td>30%</td>
<td>0.881</td>
<td>0.761</td>
</tr>
<tr>
<td>40%</td>
<td>0.947</td>
<td>0.861</td>
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<tr>
<td>50%</td>
<td>0.976</td>
<td>0.929</td>
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</table>
### Recovered and Historical Statistics

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<tr>
<th>Historical</th>
<th>Recovered</th>
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<tr>
<td>mean</td>
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</tr>
<tr>
<td>excess return</td>
<td>3.06%</td>
</tr>
<tr>
<td>sigma</td>
<td>15.30%</td>
</tr>
<tr>
<td>Sharpe</td>
<td>0.20</td>
</tr>
<tr>
<td>rf</td>
<td>5.34%</td>
</tr>
<tr>
<td>divyld</td>
<td>3.25%</td>
</tr>
<tr>
<td>ATM ivol</td>
<td></td>
</tr>
</tbody>
</table>

### Some Applications and a To Do List

- Test recovery theory by comparing the predictions it would have made at past times with what subsequently occurred
  - Use recovery theory to estimate the chances of catastrophes and booms
- Compare recovered predictions with other economic and capital market factors to find potential hedge and/or leading/lagging indicator relationships
  - This could prove valuable for asset allocation and a host of practical issues
- Extend the analysis to the fixed income markets
  - Already underway by Peter Carr and his coauthors and in joint research by myself and Ian Martin
- Finally, engage in a ‘spirited’ academic debate as to whether or not this new approach ‘fits’ the data
  - It has already started